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REPORT No. 900

# Data Reduction For The Free Flight Spark Ranges

C. H. MURPHY

DEPARTMENT OF THE ARMY PROJECT No. 503-03-001
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0108

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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ABSTRACT

The data reduction process for the Free Flight Spark Ranges is described with emphasis on recent modifications. The most important modification which is that of swerve reduction is treated in some detail. Criteria for quality of results are discussed and a table of time required for present machine data reduction routines is included.

## INTRODUCTION

One of the basic problems of the free flight range technique is that of data reduction. This is partly due to the fact that in range work the actual motion of a missile in flight is observed, and the aerodynamic forces and moments are inferred from this motion. Due to its importance for the interpretation of range firings, the cumbersome nature of the analysis actually limits the output of the ranges. For this reason a great effort has been made to refine the whole process and to make as much use of large scale computing machines as possible.

One book [1] and two reports [2] , [3] have described most of the phases of such an analysis which now is somewhat out of date. This report will attempt to describe completely the present data reduction with an emphasis on points which were not covered in previous publications.

The report will be divided into five sections:

1. Determination of atmospheric conditions and the geometrical motion.
2. Drag and roll reductions.
3. Yaw reductions.
4. Swerve reductions.
5. Criteria for quality of results.

## DETERMINATION OF ATMOSPHERIC CONDITIONS AND GEOMETRICAL MOTION

Before discussing the necessary measurements and computations we will first describe the two Free Flight Spark Ranges. A more complete description may be found in [11] .

The first range to be put into operation was the Aerodynamics Range. Through it models with body diameters or fin spans up to 57mm are launched from a variety of guns and are observed by as many as 46 spark photographic stations located over a 285' portion of the trajectory. Figure 1 shows the range as seen from the gun position. The cylinders on the left are spark boxes which provide a spark of less than one microsecond's duration. The missiles will normally pass inside the brass frames on the right and trigger the spark either electrostatically or electromagnetically. Photographic plates placed on the steel supports of each station then record the shadow of the missile directly, and indirectly by means of the mirrors. Figure 2 illustrates this arrangement. At up to twelve stations, the time intervals between spark discharge can be recorded to a least count of 5/8 microsecond. The stations are surveyed to a positional accuracy of .01 inch and an angular accuracy of 3 minutes.

The Transonic Range is shown in Figure 3. It observes 680 feet of trajectory, contains 25 spark stations and can launch missiles up to eight inches

in diameter. Its instrumentation is complicated by the introduction of a second spark and also cameras which record the shadows cast on 12' and 15' beaded motion-picture type screens. In Figure 4 the geometrical details of a spark station are shown. Due to the greater distances involved and increased light required the sparks are about two microseconds in duration. Spark interval timing pulses can be recorded at sixteen stations. The surveyed distances in the Transonic Range have an accuracy of about .01'. An angular accuracy of 6 minutes of arc is obtained.

a. Atmospheric Conditions

For theoretical considerations it is necessary to know the velocity of sound and density of air. It is attempted to keep them constant throughout the observed trajectory by air conditioning in the Aerodynamics Range and by complete insulation and heating in the Transonic Range. Before each firing the temperature, pressure, and relative humidity are measured. From the temperature it is then possible to calculate the velocity of sound in dry air, $V_{SD}$, by the relation

$$V_{SD} = 65.75 \sqrt{T} \tag{1}$$

where $V_{SD}$ is in feet/sec; T is the temperature in degrees Kelvin.

In [14] the following small correction for humidity is derived:

$$V_S = (1 + .14 \frac{P_w}{P}) V_{SD} \tag{2}$$

where $P_w$ is pressure of water vapor[1]

P is total air pressure

Due to the presence of water vapor the calculation of density is of some interest. If P is the total pressure, $P_a$ the partial pressure of water vapor, by use of Dalton's Law of Partial Pressure we can write:

$$P = P_a + P_w$$

But if we assume the perfect gas law, this can be written as

$$\rho = \rho_a + \rho_w$$

where the densities, $\rho_i$, are for the same temperature, T, but different pressures, namely P, $P_a$, $P_w$ respectively. We now refer $\rho_a$, $\rho_w$ to standard conditions $T_o$ of $0^oC$ and $P_o$ of atmospheric pressure;

1 This pressure is easily calculated by means of the relative humidity and tables of water vapor pressure for saturated air.

$$\rho = \rho_{ao} \frac{T_o}{T} \frac{P_a}{P_o} + \rho_{wo} \frac{T_o}{T} \frac{P_w}{P_o}$$

$$= \rho_{ao} \frac{T_o}{T} \frac{P-P_w}{P_o} + \rho_{wo} \frac{T_o}{T} \frac{P_w}{P_o}$$

$$= \frac{\rho_{ao}}{T} \frac{T_o}{P_o} \left[ P - (1 - \frac{\rho_{wo}}{\rho_{ao}}) P_w \right] \tag{3}$$

Equation (3) together with the known values of $\frac{\rho_{ao} \; T_o}{P_o}$ and $\frac{\rho_{wo}}{\rho_{ao}} = .3783$ now allows us to compute air density from measurements of P, T, and $P_w$.

b. Plate Measurement and Geometrical Calculations

On both ranges the z axis points down range from the gun, the x axis to the left looking downrange and y axis up. Each station has a local origin located on the intersection line of the planes of either the photographic plates for Aerodynamic Range or the screens for the Transonic Range so that its xy plane contains the sparks[1].

11" x 14" plates are employed on the Aerodynamics Range and are measured on ruled grids set in light boxes. Figure 5 shows part of a plate including a fiducial bar with three nicks. The edge of the bar is parallel to the z axis and the x,y plane is located by the nicks. Knowing the distance from the edge of the bar to the local z axis we can therefore make measurements in the local coordinate system. The actual measurements taken are the location of a reference point on the shadow's axis and the slope of the axis with respect to the fiducial bar.

On the Transonic Range the usual plate size is 4" x 5". These are usually measured on Mann Comparators (Figure 6) or a Telecomputing Telereader (Figure 7). The Telecomputing equipment, which is operated by the Measurement Analysis Branch of the Computing Laboratory, has the desirable characteristic of IBM card output. Since only positions may be measured by the Telereader, the slope of the axis is obtained by the measurement of two points on the shadow's axis. Figure 8 is a sample plate from the Transonic Range[2]. Measurements are made with respect to the crossed survey wires and are converted to distances on the screen by means of a magnification factor. These results are then transformed to the coordinate system formed by the intersection of extensions of the screen. For both ranges, therefore, the measurements can be

1 On the Aerodynamics Range both the actual spark and its virtual image in the mirror are considered.

2 Since the camera is focused on the screen, the clear image is of the shadow while the blurred image is of the missile itself.

reduced to locations of points on the shadows of the missile's axis and the slopes these shadows have with reference to a station coordinate system. The problem is, then, to derive from these data the missile's location and orientation in space at each spark station.

Since the sparks are located in each station's xy plane, they have coordinates $(c_1, c_2, 0)$ for the spark opposite the vertical plate and $(c_1', c_2', 0)$ for the other. (On the Aerodynamics Range the second spark is actually a virtual spark which is located behind the mirror). If we denote the coordinates of the shadow of the reference point $(x_R, y_R, z_R)$ by $(0, y_V, z_V)$ on the vertical plate and by $(x_H, 0, z_H)$ on the horizontal plate, points on the line between the sparks and respective shadows must satisfy the following equations:

$$\frac{x - 0}{c_1 - 0} = \frac{y - y_V}{c_2 - y_V} = \frac{z - z_V}{0 - z_V} \tag{4}$$

$$\frac{x - x_H}{c_1' - x_H} = \frac{y - 0}{c_2' - 0} = \frac{z - z_H}{0 - z_H} \tag{5}$$

Since the reference point must lie on both lines we have the following four equations for the three coordinates of this point[1]

$$x_R = x_H + R_X\, y_R \tag{6}$$

$$y_R = y_V + R_Y\, x_R \tag{7}$$

$$z_R = z_H - \frac{z_H}{c_2'}\, y_R \tag{8}$$

$$z_R = z_V - \frac{z_V}{c_1}\, x_R \tag{9}$$

where

$$R_X = \frac{c_1'}{c_2'} - \frac{1}{c_2'}\, x_H$$

$$R_Y = \frac{c_2}{c_1} - \frac{1}{c_1}\, y_V$$

1 The system is overdetermined because the lines are restricted by the assumption that they intersect. The survey and measurement accuracy may be checked by a comparison of the two values of $z_R$.

If we denote the direction cosines of the missile's axis by (m, n, p) then the equations of planes containing the missile's axis and the shadow of the reference point on the vertical and horizontal planes respectively are[1]:

$$\begin{vmatrix} x - c_1 & y - c_2 & z - 0 \\ 0 - c_1 & y_V - c_2 & z_V - 0 \\ m & n & p \end{vmatrix} = 0 \tag{10}$$

$$\begin{vmatrix} x - c_1' & y - c_2' & z - 0 \\ x_H - c_1' & 0 - c_2' & z_H - 0 \\ m & n & p \end{vmatrix} = 0 \tag{11}$$

If tan V is the slope of the shadow of the axis on the vertical plane, the point $(0, y_V + \tan V, z_V + 1)$ is a point which satisfies equation (10). If tan H is defined similarly, the point $(x_V + \tan H, 0, z_H + 1)$ lies in plane described by (11). After a few algebraic manipulations, these equations reduce to

$$m/p = \tan H - n/p\left[\frac{c_1'}{c_2'} - x_H/c_2' + \frac{z_H}{c_2'} \tan H\right] \tag{12}$$

$$n/p = \tan V + m/p\left[\frac{c_2}{c_1} - y_V/c_1 + \frac{z_V}{c_1} \tan V\right] \tag{13}$$

$$\text{and } m^2 + n^2 + p^2 = 1 \tag{14}$$

---

1 These determinants are the scalar triple products of the vectors $(x - c_1, y - c_2, z - 0)$, $(0 - c_1, y_V - c_2, z_V - 0)$, (m, n, p) and the vectors $(x - c_1', y - c_2', z - 0)$, $(x_H - c_1', 0 - c_2', z_H - 0)$, (m, n, p) respectively. Since scalar triple products represent the volumes of the boxes formed with the three vectors, these quantities will be zero only when the point (x, y, z) lies in the plane determined by the vector (m, n, p) and either the vector $(0 - c_1, y_V - c_2, z_V - 0)$ or the vector $(x_H - c_1', 0 - c_2', z_H - 0)$.

From (12) - (14) m, n, and p may be determined. If L denotes the distance between the reference point and the center of mass ($x_{cm}$, $y_{cm}$, $z_{cm}$), then

$$
\begin{aligned}
x_{cm} &= x_R + mL + x_i \\
y_{cm} &= y_R + nL + y_i \qquad (15) \\
z_{cm} &= z_R + pL + z_i
\end{aligned}
$$

where ($x_i$, $y_i$, $z_i$) are the coordinates of the local origin of the ith station relative to a fixed origin. The computations described by equations (6) - (15) have been coded for the Ordvac and the IBM CPC for measurements obtained by Mann Comparator or light box. At the present time the card output of the Telereader can be processed by only the Ordvac.

Finally it is necessary to compute the two components of the yaw from the direction cosines. The yaw angle measured in range work is defined to be the angle from the tangent to the trajectory to the missile's axis. For the flat trajectories encountered in spark range work and, for small yaws this can be done by the use of the simple relations:

$$\lambda_H = m - \frac{dx}{dz} \qquad (16)$$

$$\lambda_V = n - \frac{dy}{dz} \qquad (17)$$

where $\lambda_H$ is horizontal component of yaw in radians

$\lambda_V$ is vertical component of yaw in radians

$\frac{dx}{dz}$ is "horizontal slope" of trajectory

$\frac{dy}{dz}$ is "vertical slope" of trajectory

For hand reductions $\frac{dx}{dz}$ and $\frac{dy}{dz}$ may be computed by differencing values of x and y between successive stations. As coded for the Ordvac the stations are divided into four or five groups, and for each group x and y are fitted by a quadratic function of z. From these resulting equations $\frac{dx}{dz}$ and $\frac{dy}{dz}$ may be obtained.

## DRAG AND ROLL REDUCTIONS[1]

### a. Drag Reduction

The drag force as measured along the trajectory is defined to be $\rho d^2 u_1^2 K_D$, where $\rho$ is the air density, d the missile's diameter, $u_1$ the component of the missile's velocity, u, resolved along its axis of symmetry, and $K_D$ its drag coefficient. If we assume a flat trajectory directed along the z axis, $u_1 \doteq u = \frac{dz}{dt}$ for small yawing motion, and

$$m\dot{u} = -\rho d^2 u^2 K_D \tag{18}$$

If we define u' as $\frac{du}{dp}$ where $p = \frac{z}{d}$ is non- dimensional

$$\dot{u} = \frac{du}{dp}\frac{dp}{dz}\frac{dz}{dt} = \frac{u'u}{d} \tag{19}$$

and[2]

$$\frac{u'}{u} = -\left(\frac{\rho d^3}{m}\right)K_D \equiv -J_D \tag{20}$$

Since distance, not time, is a more fundamental and convenient measurement in range work, the distance is taken to be the independent variable. It has been found that a cubic equation in distance fits the timing data quite well and for this equation we have

$$t = a_o + a_1 p + a_2 p^2 + a_3 p^3 \tag{21}$$

$$u = \frac{dz}{dt} = \frac{\frac{dz}{dp}}{\frac{dt}{dp}} = \frac{d}{t'}$$

$$= \frac{d}{a_1 + 2a_2 p + 3a_3 p^2} \tag{22}$$

$$K_D = -\left(\frac{m}{\rho d^3}\right)\frac{u'}{u}$$

$$= \left(\frac{m}{\rho d^3}\right)\frac{2a_2 + 6a_3 p}{a_1 + 2a_2 p + 3a_3 p^2} \tag{23}$$

---

1 The drag reduction is also described in [1], [2], [3], and [4] while a detailed description of the roll reduction is in [10].

2 K's will always be related to J's by $\frac{\rho d^3}{m} K_i = J_i$.

From equations (22), (23) and the velocity of sound we can compute the Mach number and its associated drag coefficient. For the greatest accuracy we normally evaluate both at the center of the observed trajectory.

Equation (21) is written on the assumption that t can be represented by a cubic in z. It is known, however, that $K_D$ is a function of $\delta^2$, the squared magnitude of the yaw, and M, the Mach number. Although the Mach number variation on the range is usually small, the yaw can undergo large changes. Therefore, a yaw drag reduction was formulated by E. J. McShane [1]. We will describe a modification of this reduction. First the assumption is made that $J_D$ is linearily dependent on $\delta^2$ and M:

$$J_D = J_{D_o} + \delta^2 J_{D_{\delta^2}} + \left(\frac{u - u_o}{V_S}\right) J_{D_M} \tag{24}$$

where $J_{D_o}$, $J_{D_{\delta^2}}$ and $J_{D_M}$ are constants.

Substituting (24) in (20) and integrating,

$$\frac{u_o}{u} = e^{\int_o^p J_D \, dp}$$

$$\doteq 1 + J_{D_o} p + J_{D_{\delta^2}} \int_o^p \delta^2 \, dp + \frac{1}{V_S} J_{D_M} \int_o^p (u - u_o)\, dp + \frac{1}{2}\left[\int_o^p J_D \, dp\right]^2 \tag{25}$$

We now replace the quadratic term by $\frac{1}{2} \bar{J}_D{}^2 p^2$ where $\bar{J}_D$ is an average value of the drag coefficient. Using the fact that $\frac{1}{u} = \frac{dt}{dz} = \frac{dt}{dp}\left(\frac{1}{d}\right)$, integrating and rearranging,

$$t = t_o + \frac{d}{u_o} p + \frac{d}{u_o} J_{D_o} \frac{p^2}{2} + \frac{d}{u_o} \bar{J}_D^2 \frac{p^3}{6} + \frac{d}{u_o} J_{D_{\delta^2}} I_1(p) + \frac{d^2}{u_o V_S} J_{D_M} I_2(p) \tag{26}$$

$$I_1(p) = \int_0^p \int_0^p \delta^2 \, dp \, dp$$

$$I_2(p) = \int_0^p \int_0^p \left(\frac{u - u_o}{d}\right) dp \, dp$$

Assuming that u can be approximated by (22),

$$\frac{u - u_o}{d} = \frac{1}{a_1} \left( \frac{1}{1 + \frac{2a_2}{a_1} p + \frac{3a_3}{a_1} p^2} - 1 \right)$$

$$\doteq - \frac{2a_2}{a_1^2} p$$

$$I_2(p) \doteq - \frac{a_2}{a_1^2} \frac{p^3}{3}$$

For convenience we select $\bar{J}_D$ to be the value supplied by the standard drag reduction, namely $\frac{2a_2}{a_1}$ . Therefore

$$t = t_o + \frac{d}{u_o} p + \frac{d}{u_o} J_{D_o} \frac{p^2}{2} + \frac{d}{u_o} \bar{J}_D (J_D - \bar{M} \bar{J}_{D_M}) \frac{p^3}{6} + \frac{d}{u_o} J_{D_{\delta^2}} I_1(p) \tag{26'}$$

where $\bar{M} = \frac{d}{V_S a_1}$ is the average Mach number since p is selected to be zero at the middle of observed trajectory.

It now remains to evaluate $I_1(p)$.

In [1] or [5], it is shown that for a symmetrical missile the total yaw $\lambda$ can be written as

$$\lambda = \lambda_H + i\lambda_V = K_1 e^{(-\alpha_1 + i\phi_1')p} + K_2 e^{(-\alpha_2 + i\phi_2')p} + \lambda_R \tag{27}$$

where $K_i$ are complex constants

$\alpha_i$ are real constants

$\phi_i'$ are real linear functions of p

$\lambda_R$ is the "yaw of repose" and is determined by a yaw reduction as described in the next section

If $\lambda_R$ is small,

$$\delta^2 = \lambda \bar{\lambda} = K_{10}^2 e^{-2\alpha_1 p} + K_{20}^2 e^{-2\alpha_2 p} + 2K_{10}K_{20} e^{-(\alpha_1 + \alpha_2)p} \cos\left[(\phi_2' - \phi_1')p + (\phi_{20} - \phi_{10})\right] \tag{28}$$

where $K_i = K_{io} e^{i\phi_{io}}$

If (28) is placed in the definition of $I_1$ and the indicated integrations are performed, there results[1]:

$$I_1(p) = K_{10}^2 \left[ \frac{e^{-2\alpha_1 p} - 1 + 2\alpha_1 p}{(2\alpha_1)^2} \right] + K_{20}^2 \left[ \frac{e^{-2\alpha_2 p} - 1 + 2\alpha_2 p}{(2\alpha_2)^2} \right]$$

$$+ \frac{2K_{10}K_{20}}{r^2} \left\{ e^{-(\alpha_1 + \alpha_2)p} \cos \left[ (\phi_2' - \phi_1')p + \gamma \right] - \cos \gamma - p\, r \cos (\gamma + \eta) \right\} \qquad (29)$$

where $r = \sqrt{(\phi_2' - \phi_1')^2 + (\alpha_1 + \alpha_2)^2}$

$$\gamma = \phi_{20} - \phi_{10} - 2\eta$$

$$\eta = \text{arc cos} \frac{-(\alpha_1 + \alpha_2)}{r} = \text{arc sin} \frac{\phi'_2 - \phi'_1}{r}$$

The procedure for yaw drag reduction is first to perform a standard drag reduction in order to obtain values of $\bar{J}_D$ and $\bar{M}$ which together with an estimated $J_{D_M}$ determine the cubic coefficient. A least squares is then run for the remaining unknown coefficients, $t_o$, $\frac{d}{u_o}$, $\frac{d}{u_o} J_{D_o}$, and $\frac{d}{u_o} J_{D_{\delta^2}}$. If $J_{D_M}$ is not fixed, a least squares fit of five parameters can be attempted. This latter procedure has been ineffective up to the present time[2].

The variation of drag with yawing motion for a group of identically shaped models fired at about the same Mach number can be determined rather simply, however, from the standard drag reduction. We assume that their individual drag coefficients are average $J_D$'s, over the length of observed trajectory. Eq. (24) can be written in the form

$$J_D = J_{D_o} + \overline{\delta^2}\, J_{D_{\delta^2}} \qquad (24')$$

1 The cosine term is integrated as though $\phi'_i$ were constants.

2 This yaw drag reduction has been coded for the Bell Relay Computer by J. Schmidt and L. Schmidt.

where $\overline{\delta^2} = \frac{1}{p_L - p_F} \int_{p_F}^{p_L} \delta^2 \, dp$ (mean squared yaw)

$p_F$ is p coordinate of missile at the first timing station

$p_L$ is p coordinate of missile at the last timing station

The value of $\overline{\delta^2}$ is a by-product of the first integration involved in obtaining Eq. (29):

$$\overline{\delta^2} = \frac{1}{2(p_L - p_F)} \left[ K_{10}^{2} \left( \frac{e^{-2\alpha_1 p_F} - e^{-2\alpha_1 p_L}}{\alpha_1} \right) + K_{20}^{2} \left( \frac{e^{-2\alpha_2 p_F} - e^{-2\alpha_2 p_L}}{\alpha_2} \right) \right]$$

where the integral of the cosine term is neglected for the usual large values of $p_L - p_F$ encountered. The drag coefficients for each round are then plotted against the corresponding values of the mean squared yaw. A line is fitted to these points and good determination of $K_{D_o}$ and $K_{D_{\delta^2}}$ for the common Mach numbers can be made.

b. Roll Reduction

The equation of rolling motion for a missile with rotational symmetry may be written in the form [10] :

$$\nu' = D\nu + k_1^{-2} J_{A_o} = \hat{D}\,\nu$$

$$\text{where } \nu \equiv \theta' = \frac{\omega_1 d}{u} \qquad (30)$$

$\theta$ is roll angle

$\omega_1$ axial angular velocity

$D = J_D - k_1^{-2} J_A; \quad \hat{D} = D + \frac{k_1^{-2} J_{A_o}}{\nu}$

$k_1^{-2} = \frac{md^2}{A}$ ($k_1$ is axial radius of gyration in calibers)

m = mass

A = axial moment of inertia

and $K_A$ and $K_{A_o}$ are defined by the relation: axial aerodynamic moment = $\rho d^3 u^2 \left[ K_{A_o} - \nu K_A \right]$

The solution of (30) can be easily obtained and is

$$\theta = B + sp + Ae^{Dp} \tag{31}$$

where $s = -\frac{k_1^{-2} J_{A_0}}{D}$ (steady state roll per caliber).

For range work the roll angle may be measured by means of two pins of different shape, in the base of the missile (see Fig. 8). The location of these pins in space may be calculated by use of a slightly modified center of mass reduction coding and the orientation of the vector between the pins then provides the roll angle.

If D is sufficiently small[1] the exponential may be expanded as a cubic in p and the standard drag reduction coding which is available both on the Ordvac and the Bell Computer can be used to fit θ as a function of z. In general, however, a new reduction procedure is required. Furthermore since the unknown coefficient D appears in a non-linear fashion the iterative method of differential corrections is needed.

In order to start the iterations a set of initial values is a prerequisite. To do this we differentiate (31) and use the result to eliminate the exponential:

$$\theta = (B - \frac{s}{D}) + sp + \frac{1}{D}\theta' \tag{32}$$

$\theta'$ can be computed by numerical differentiation using first differences and equation (32) fitted by the routine least squares since it is linear in unknowns $(B - \frac{s}{D})$, s, and $\frac{1}{D}$. From these B, s, D follow and by use of (31) for a particular station, A is determined.

From equation (31) a relation in the differential corrections of these values can be derived:

$$\Delta\theta = \theta_{observed} - \theta_{computed} = \Delta B + p\Delta s + e^{Dp}\Delta A + Ap\, e^{Dp}\Delta D \tag{33}$$

θ is computed from (31) by use of the initial computed values of A, B, s, D or the values obtained from the preceding iteration. Since (33) is linear in the correction, ordinary least squares apply and a set of corrected coefficients is obtained. If the initial values are close enough, the process will converge. Most rounds require no more than two iterations. The complete roll reduction is coded for the Ordvac while the iterative process of (33) is coded for the Bell Computer.

1 This is the case for bodies of revolution.

In [16] the variation of $K_A$ and $K_{A_o}$ with Mach number is considered. There it is shown that $K_A$ can be quite well approximated by an inverse linear function of Mach number. For simplicity we will make the quite reasonable approximation that D itself is such a function.

$$\therefore \; D = (a + bM)^{-1}$$

Now from Eq. (22) it is easy to show that a good approximate relation between Mach number, M, and position on range, p, is

$$M = M_o(1 - J_D p)$$

From these equations we have

$$D = \frac{D_o}{1 + \gamma p}$$

where $D_o = (a + bM_o)^{-1}$ (value of D at $p = 0$)

$$\gamma = \frac{-J_D}{1 + \frac{a}{bM_o}}$$

The further assumption that the ratio of $k_1^{-2} J_{A_o}$ to D is a linear function of M or p must now be made.

$$k_1^{-2} J_{A_o} = - D(s_o + s'p)$$

From these assumptions the following revised Eq. (30) results

$$\nu' = D(\nu - s_o - s'p)$$

$$= \frac{D_o}{1 - \gamma p}(\nu - s_o - s'p) \qquad (30')$$

This can be integrated to

$$\theta = B + sp + Cp^2 + Ae^{f(p)} \qquad (31')$$

where $f(p) = \frac{\gamma + D_o}{\gamma} \ln(1 + \gamma p) = Ep(1 - \frac{\gamma p}{2} + \frac{\gamma^2 p^2}{3} - \frac{\gamma^3 p^3}{4} \ldots)$

$$s = s_0 + \frac{s'}{D_0 - \gamma}$$

$$C = \frac{s'}{2}\left(\frac{D_0}{D_0 - \gamma}\right)$$

$$E = \gamma + D_0$$

By use of Eq. (31') we can now modify Eqs. (32) and (33).

$$\theta = (B - \frac{s}{f'}) + (s - \frac{2C}{f'})p + Cp^2 + \frac{1}{f'}\theta' \qquad (32')$$

where $f' = E\left[1 - \gamma p + (\gamma p)^2 - (\gamma p)^3 \ldots\right] \doteq E$

$$\Delta\theta = \Delta B + p\Delta s + p^2\Delta C + e^{f(p)}\Delta A + Ae^{f(p)}p(1 - \frac{\gamma p}{2} + \frac{\gamma^2 p^2}{3} - \ldots)\Delta E$$

$$-Ae^{f(p)}Ep^2\left[\frac{1}{2} - \frac{2\gamma p}{3} + \frac{3(\gamma p)^2}{4} - \ldots\right]\Delta\gamma \qquad (33')$$

The procedure is quite similar to the regular roll reduction. First B, s, C , and E are calculated from Eq. (32') by a least squares fit. These are placed in Eq. (31') together with $\theta$ and p at the station for which $p = 0$ and B is computed. Next $\gamma$ is either estimated or computed from Eq. (31') for a station at which p is large. Finally Eq. (33') is employed in as many iterations as necessary for complete convergence. The problem as it is being coded for the ORDVAC will have the option of fixing $\gamma$ ($\Delta\gamma = 0$) or allowing it to vary. This reduction then will be either a five or six unknowns problem. Recent experience with small light finned missiles definitely shows the need for this modified reduction.

## YAW REDUCTIONS[1]

For missiles possessing angles of rotational symmetry less than 180° and planes of mirror symmetry, the definition of the linearized force and moment system is well known [1], [9]. If the requirement of mirror symmetry is relaxed slightly[2] we have the spin producing moment $J_{A_0}$ of equation (30), [10] If in addition to this we assume either a slight configurational asymmetry or slight mass asymmetry [13], the transverse force and moment definitions receive a constant missile-attached force and moment increment. In the theoretical development of [9] the usual coordinate system is orientated so that the one axis lies along the missile's axis of symmetry, the two axis is in the horizontal plane pointing to the right and the three axis is determined by the right hand rule. This selection of the 2 and 3 axes is the exact reverse of the orientation of the ranges' x and y axes. Since, however, the yaw

1 In order to understand fully the next two sections, they should be read in conjunction with [9].

2 In other words the assumption is made that the differential canting of fins of a finned missile affects only the axial moment.

as defined in [9] is measured from the missile's axis to the tangent to the trajectory, the reverse of the range definition, we have $\lambda_2 = \lambda_H$ and $\lambda_3 = \lambda_V$ where $\lambda_2$, $\lambda_3$ are the yaw components of [9]. The aerodynamics force and moment acting perpendicular to the missile's axis can now be defined as:

$$F_2 + iF_3 = \rho d^2 u_1^{\,2} \left[(-K_N + i\nu K_F)\lambda + (\nu K_{XF} + iK_S)\mu + K_{N_\epsilon}\lambda_\epsilon \, e^{i\theta}\right] \tag{34}$$

$$M_2 + iM_3 = \rho d^3 u_1^{\,2} \left[(-\nu K_T - iK_M)\lambda + (-K_H + i\nu K_{XT})\mu + iK_{M_\epsilon}\lambda_\epsilon \, e^{i\theta}\right]$$

where $\theta = \int_0^p \nu dp + \theta_o$ (roll angle with respect to force due to asymmetry)

$$\nu = \frac{\omega_1 d}{u_1}$$

$(\omega_1, \omega_2, \omega_3)$ angular velocity vector

$\lambda = \lambda_2 + i\lambda_3$ (complex yaw)

$\mu = \dfrac{\omega_2 d}{u_1} + i\,\dfrac{\omega_3 d}{u_1}$ (non-dimensional angular velocity)

$\lambda_\epsilon$ magnitude of asymmetry angle

$K_N$ Normal force coefficient

$K_F$ Magnus force coefficient

$K_S$ Damping force coefficient

$K_{XF}$ Magnus force coefficient due to cross spin

$K_T$ Magnus moment coefficient

$K_M$ Overturning (or righting) moment coefficient

$K_H$ Damping moment coefficient

$K_{XT}$ Magnus moment coefficient due to cross spin

$K_{N_\epsilon}$ Normal force coefficient due to small asymmetry

$K_{M_\epsilon}$ Moment coefficient due to small asymmetry[1]

[1] In the case of mass asymmetry only the product $K_{M_\epsilon}\lambda_\epsilon$ has meaning and can be computed from the physical measurements of the projectile. This remark, of course, also applies to the product $K_{\overline{N_\epsilon}}\lambda_\epsilon$.

It is easy to see that the exponential coefficients of $K_{N_\epsilon}$ and $K_{M_\epsilon}$ perform the task of insuring that the force and moment associated with the asymmetry be missile-attached.

The equations of motion which can be obtained from the above definitions for flat trajectories are:

$$\lambda'' + (H - i\bar{\nu})\lambda' + \left[-M - i\bar{\nu}T\right]\lambda = G + J_\epsilon e^{i\theta} \tag{35}$$

$$\bar{\nu}' = \hat{D}\,\bar{\nu}$$

where

$$H = J_L - J_D + k_2^{-2} J_H\ ;\ J_L = J_N - J_D$$

$$\bar{\nu} = \frac{A}{B}\,\nu$$

B transverse moment of inertia

$$M = k_2^{-2}\left[J_M + \nu^2 k_1^2\ J_F\right] \doteq k_2^{-2} J_M$$

$$T = J_L - k_1^{-2} J_T$$

$$G = \frac{igd}{u^2}\left[J_D + k_2^{-2}\ J_H - i\bar{\nu}\right]$$

$$J_\epsilon = \left[i\nu\left(1 - \frac{A}{B}\right) J_{N_\epsilon} - k_2^{-2} J_{M_\epsilon}\right]\lambda_\epsilon$$

$$k_2^{-2} = \frac{md^2}{B}$$ ($k_2$ is transverse radius of gyration in calibers).

The solutions for these equations of motion can be written for the case of slowly changing spin as

$$\lambda = K_1 e^{-\alpha_1 p + i(\phi_{10}' p + \phi_1'' \frac{p^2}{2})} + K_2 e^{-\alpha_2 p + i(\phi_{20}' p + \phi_2'' \frac{p^2}{2})} + \lambda_R + K_3 e^{i\theta} \tag{36}$$

where[1]

$$-\alpha_1 p + i(\phi_{10}' p + \phi_1'' \frac{p^2}{2}) = 1/2\int_0^p \left\{-H + i\bar{\nu} - \hat{D}\epsilon_1 + \left[-\bar{m} + 2i\bar{\nu}\,(2T - H - \hat{D})\right]^{1/2}\right\}dp$$

$$-\alpha_2 p + i(\phi_{20}' p + \phi_2'' \frac{p^2}{2}) = 1/2\int_0^p \left\{-H + i\bar{\nu} - \hat{D}\epsilon_1 - \left[-\bar{m} + 2i\bar{\nu}\,(2T - H - \hat{D})\right]^{1/2}\right\}dp$$

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[1] See Eqs. (8) and (11) of [9]. For the relation defining $K_3$ see [13].

$$\lambda_R = \frac{-Zd}{Ku^2}\left[\bar{\nu} - i(-J_D - k_2^{-2} J_H + \bar{\nu}^2 \frac{T}{M})\right]\left[1 + \frac{\bar{\nu}^2 T}{M}\right]^{-1}$$

$$\epsilon_1 = \frac{\bar{\nu}^2\left[\bar{m} + 2(2T-H-\hat{D})^2\right]}{\bar{m}^2 + 4\bar{\nu}^2 (2T-H-\hat{D})^2}$$

$$\bar{m} = \bar{\nu}^2 - 4M - H^2 .$$

$K_1$ and $K_2$ are complex constants depending on initial conditions

$$K_3 = \frac{J_\epsilon}{\left[-\alpha_1 + i(\phi_1{}' - \nu)\right]\left[-\alpha_2 + i(\phi_2{}' - \nu)\right]} = K_{30}\, e^{i(\phi_{30} - \theta_o)};\ K_{30} \text{ real.}$$

If we differentiate and add the definitions for the exponents in Eq. (36) and equate the real and imaginary parts,[1]

$$\bar{\nu} = \phi_1{}' + \phi_2{}' \quad \text{where } \phi_i{}' = \phi_{io}' + \phi_i{}'' p \tag{37}$$

$$H = (\alpha_1 + \alpha_2) - \hat{D}\,\epsilon_1 \tag{38}$$

Next we differentiate and subtract the definitions

$$\left[-\bar{m} + 2i\bar{\nu}\,(2T-H-\hat{D})\right]^{1/2} = -(\alpha_1 - \alpha_2) + i(\phi_1' - \phi_2')$$

$$\therefore \bar{m} = (\phi_1' - \phi_2')^2 - (\alpha_1 - \alpha_2)^2 \tag{39}$$

$$\therefore \bar{\nu}\,(2T-H-\hat{D}) = -(\alpha_1 - \alpha_2)\,(\phi_1' - \phi_2') \tag{40}$$

Inserting the definition of $\bar{m}$ in (39) and using (37) and (38),

$$M = \phi_1' \cdot \phi_2' - \alpha_1\alpha_2 + \frac{\epsilon_1}{2}\,\hat{D}\,(\alpha_1 + \alpha_2 - \frac{\epsilon_1}{2}\,\hat{D}). \tag{41}$$

Equation (40) can be solved for T:

$$T = -1/2\left[(\alpha_1 - \alpha_2)\,\frac{\phi_1' - \phi_2'}{\phi_1' + \phi_2'} - H-\hat{D}\right] \tag{42}$$

[1] Although, as we shall see, $\hat{D}$ can be computed from the yaw reductions, the resulting values are sometimes quite bad due to experimental limitations. For use in Eqs. (38), (41) and (42), $\hat{D}$ should either be estimated or directly measured by a roll reduction.

It remains to compute $\epsilon_1$ from its definition. But this can be done by means of equations (36), (39) and (40).

$$\epsilon_1 = \frac{(\phi_1' + \phi_2')^2 \left[(\phi_1' - \phi_2')^2 - (\alpha_1 - \alpha_2)^2\right] + 2(\alpha_1 - \alpha_2)(\phi_1' - \phi_2')}{2 \cdot \left[(\phi_1' - \phi_2')^2 + (\alpha_1 - \alpha_2)^2\right]^2}$$

$$\doteq \left[\frac{\phi_1' + \phi_2'}{\phi_1' - \phi_2'}\right]^2 . \tag{43}$$

$\alpha_1$, $\alpha_2$, $\phi_1'$, $\phi_2'$ are quantities which are obtained in the curve fitting process to be described shortly. By means of equations (37), (38), (41), (42), and (43), the spin $\bar{\nu}$, and three combinations of aerodynamic coefficients M, H, and T may be calculated. Although $K_M$ can be directly obtained from M, an independent evaluation of $K_L$ is required in order to obtain $K_H$ and $K_T$ from H and T respectively. The variation of the turning rates is caused physically by the change in $\bar{\nu}$ and M during flight. Using equations (30), (37), and (41) we have

$$\hat{D} = \frac{\bar{\nu}'}{\bar{\nu}} = \frac{\phi_1'' + \phi_2''}{\phi_1' + \phi_2'}, \tag{44}$$

$$M' = \phi_1' \cdot \phi_2'' + \phi_2'' \cdot \phi_1' . \tag{45}$$

Symmetrical (Epicyclic) Yaw Reduction

We will first discuss the reduction process for a symmetrical missile. ($K_{M_\epsilon}\lambda_\epsilon = K_{N_\epsilon}\lambda_\epsilon = K_3 = 0$). This yields a rosette or epicyclic type yawing motion. Fig. 9 shows the yawing motion of a spinning body of revolution with negligible yaw of repose and Fig. 10 displays that of a rapidly spinning finned missile.

The first step in this reduction is the calculation of the yaw of repose. For velocities greater than 600 ft/sec.[1], a good approximation from its definition can be obtained from equation (36) and is $\lambda_R = \frac{-gd\bar{\nu}}{Mu^2}$ . All the quantities in the above relation are either known quantities or may be quite easily estimated. Of course once initial values of $\phi_i'$ are obtained, equations (37) and (41), neglecting the effect of $\alpha$, may be used. The yaw of repose is real and negative for positive spins which means that the missiles will tend to point slightly to the right of the trajectory.[2]

1 Range firings are usually restricted to velocities greater than this value.

2 It should be remembered that the positive real axis has been selected to point to the left.

This computed value of $\lambda_R$ is then subtracted either numerically or graphically from the measured $\lambda_H$'s and the resulting yaw values plotted in groups of five or six stations. For these groups, which cover about 15% of the total length of observed flight, the effects of damping and change in turning rates are assumed to be negligible. $\left(\alpha_i = \phi_i'' = 0\right)$ The points should then satisfy equations of the form

$$\lambda - \lambda_R = K_{10}\, e^{i\left[\phi_{10} + \phi_{10}'\,(p-p_o)\right]} + K_{20}\, e^{i\left[\phi_{20} + \phi_{10}'\,(p-p_o)\right]} \tag{46}$$

where $K_i = K_{io}\, e^{i\phi_{io}}$; $K_{io}$ real

$p_o$ is value of p for middle station of group. Multiplying by $e^{-i\phi'(p-p_o)}$

$$e^{-i\phi'(p-p_o)}(\lambda - \lambda_R) = K_{10}\, e^{i\left[\phi_{10} + (\phi_{10}' - \phi')(p-p_o)\right]} + K_{20}\, e^{i\left[\phi_{20} + (\phi_{20}' - \phi')(p-p_o)\right]} \tag{47}$$

This can be done quite easily by use of a compass since this multiplication is equivalent to rotation of the yaw angle at p through the angle $-\phi'\,(p-p_o)$. Should $\phi'$ be so guessed that it is quite close to the local slow rate $\phi_{20}'$ the above equation reduces to the equation of a circle:

$$e^{-i\phi_{20}'\,(p-p_o)}\,(\lambda - \lambda_R) = K_{10}\, e^{i\left[\phi_{10} + (\phi_{10}' - \phi_{20}')(p-p_o)\right]} + K_{20}\, e^{i\phi_{20}} \tag{48}$$

This trial and error process[1] can usually be done quite quickly and the completed solution is shown in Fig. 11. Note that $\phi_{10}'$ can be obtained from the position of the points on the circles. If the $\phi_{io}$'s obtained for adjacent groups with due care for multiples of 360° are considered, somewhat more accurate values of $\phi_i'$'s may be computed for the midpoint of the interval between them. If these values for all of the groups are plotted against p, estimates of $\phi_i''$ can be made by use of a straight line fit. Finally the $K_{io}$'s are plotted on semi-log paper against p and $\alpha_i$ can be determined from the slope of a fitted straight line.

From the above brief description, it can be seen that first values of the four initial constants $K_{io}$, $\phi_{io}$'s for the range origin, and the six parameters $\alpha_i$, $\phi_{io}'$ and $\phi_i''$ may be determined. It is now necessary to apply

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1 An interesting variation of this process which makes use of an analogue computer is described in [7].

the method of differential corrections which was first described for the roll reduction. The equation for the differential correction and the resulting normal equations are given in [3] and [8].[1] This ten unknown yaw reduction has been coded for the Bell Relay Computer, Eniac, Edvac and Ordvac.

If a roll reduction has been done, much additional information is available for the yaw reduction. The $\hat{D}$ which appears in equations (38), (41) and (42) is always computed from the drag and roll reductions or estimated from experience with the configuration under consideration and is not obtained from Eq. (44). The two values are compared in order to give an indication of the quality of the yaw reduction. Insertion of known values of $\hat{D}$ and $\nu$ would eliminate one $\phi_i'$ and one $\phi_i''$. The requirement that $M' = 0$ would then eliminate the other $\phi_i''$. Finally an inspection of equation (36) shows that $\alpha_1 = \alpha_2$ when $\nu = 0$. Thus we see that the number of unknowns can vary from six to ten. As a result of this fact three variations of the basic ten unknowns yaw reduction have been coded for the Bell Relay Computer. They are:

1. eight unknowns[2] reduction which uses fixed values of $\phi_i''$ as obtained from equations (44) and (45) and initial values of $\phi_i'$.

2. seven unknowns reduction which uses fixed values of $\phi_i''$ and the additional restriction that $\phi_1' + \phi_2' = \bar{\nu}$ is known.

3. six unknowns reduction which is based on the assumption of zero spin and constant moment coefficient ($M' = 0$). For this reduction $\phi_1' + \phi_2' = 0$, $\alpha_1 = \alpha_2$, and $\phi_1'' = \phi_2'' = 0$.

For certain rounds these special reductions have proven to be superior to the basic ten unknowns.

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1 [3] uses the triangular square root method of matrix inversion while [8] advocates two methods which make use of the extensive symmetry of the normal equations. For considerations of machine time we have found that most of the time of computation is devoted to input, output, and formation of normal equations. As a result of this experience most of our recent work has used essentially the Gaussian elimination method of inversion. It has also been found that a floating binary point routine as described in [12] is of great value and does not greatly increase machine time requirements.

2 When a variation of the eight unknowns yaw reduction was coded for the CPC at the Naval Ordnance Laboratory, use was made of the symmetry by means of performing all operations in the arithmetic of complex numbers. This was possible because this reduction can be considered as a problem in four complex unknowns.

Asymmetrical (Tricycle) Yaw Reduction

Since it is extremely difficult to reduce the yawing motion of an asymmetrical missile without knowledge of the rolling motion, we will only consider the reduction of rounds for which a roll reduction has been performed. It should be emphasized that equation (36) is exactly correct only when $\nu$ is constant. It is assumed that for small variations of $\nu$, when it does not get close to $\phi_1'$, the relation is still valid. When $\nu$ varies near $\phi_1'$, resonance occurs and equation (36) clearly is not valid.[1] From the roll reduction we know the change in $\phi_3$ but not $\phi_{30}$. The problem then is to compute $\phi_{30}$ and $K_{30}$.

First all of the yaw points are plotted and rotated through angle $-(\theta - \theta_o)$. This yields[2]

$$\lambda\, e^{-(\theta - \theta_o)i} = K_1\, e^{(-\alpha_1 + i\phi_1')p - i(\theta - \theta_o)} + K_2\, e^{(-\alpha_2 + i\phi_2')p - i(\theta - \theta_o)} + K_{30} e^{i\phi_{30}} \qquad (49)$$

Since $\theta$ is linear or very nearly linear, the result is an epicyclic motion about the point $K_{30}\, e^{i\phi_{30}}$ (Figures 12, 13, 14, illustrate this process). It is now easy to determine $K_{30}$ and $\phi_{30}$ since they are the polar coordinates of the center of this epicyclic motion. The origin is moved to this point and the points rotated back through the angle $\theta - \theta_o$. It is now possible to perform the usual epicyclic reduction and so obtain the remaining parameters.

The tricyclic yaw differential corrections routine has been coded in two forms on the Bell Relay Computer and is now being coded for the Ordvac. The reduction as coded for the Ordvac will be a nine unknowns problem formed by combining the seven unknowns epicyclic yaw reduction with the two parameters $K_{30}$, $\phi_{30}$ of the tricycle arm. The two Bell Relay Computer versions are eight and ten unknowns problems and were formed by combining the two tricycle parameters with the six unknowns and eight unknowns epicyclic yaw reductions respectively.

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1 In order to reduce recent firings by J. D. Nicolaides and L. C. MacAllister for asymmetrical rounds which go through resonance, use is being made of the Exterior Ballistics Laboratory's Analog Computer in a fashion similar to that described in [7]. Work is being done on this by J. Schmidt.

2 This method was suggested by L. C. MacAllister [15]. $\lambda_R$ for most tricyclic models is extremely small and so is neglected.

## SWERVE REDUCTIONS

The last bit of dynamic information which can be obtained from a free flight range firing lies in the lateral motion of the missile. The equation of motion, which the x and y coordinates of the center of mass of the missile must satisfy, may be written in the form:

$$m(\ddot{x} + i\ddot{y}) = (F_x + iF_y) - img - ma_C \qquad (50)$$

where $F_x$ is x component of the aerodynamic force

$F_y$ is y component of the aerodynamic force

$g$ is acceleration due to gravity[1]

$a_C$ is the Coriolis acceleration

If we resolve the aerodynamic force along the x and y axes, remembering that the small angle between the 1 axis and z axis is $\lambda + \frac{\dot{x} + i\dot{y}}{u}$ and that the 2-3 axes are the negatives of the xy axes when this angle is zero:

$$F_x + iF_y = -(F_2 + iF_3) + F_1 \left(\lambda + \frac{\dot{x} + i\dot{y}}{u}\right) \qquad (51)$$

But $F_1$ is defined by the relations:

$$F_1 = -\rho u_1^2 d^2 K_{DA} = m\dot{u}_1 \qquad u_1 \doteq u \qquad K_{DA} \doteq K_D \qquad (52)$$

In [1] and [9] the following equation is derived for the symmetrical missile:

$$\lambda' - i\mu = (-J_L + i\nu J_F)\lambda + (\nu J_{XF} + iJ_S)\mu + \gamma \doteq 0 \qquad (53)$$

where $\gamma$ is a small contribution of gravity.

The insertion of the term $J_{N_\epsilon}\lambda_\epsilon e^{i\theta}$ in Eq. (53) for the asymmetric missile does not effect the approximation $\lambda' - i\mu \doteq 0$.

Using relations (34) (52) and (53) and converting from $K_i$'s to $J_i$'s where necessary we can rewrite (51) in the form:

$$F_x + iF_y = \frac{mu^2}{d}\left[(J_L - i\nu J_F)\lambda + (-J_S + i\nu J_{XF})\lambda' - J_{N_\epsilon}\lambda_\epsilon e^{i\theta}\right] + \frac{m\dot{u}}{u}(\dot{x} + i\dot{y}) \qquad (54)$$

---

1 The standard value of g at the Aberdeen Proving Ground is 32.152 ft/sec.$^2$.

The usual change of independent variable is now made

$$x'' + iy'' = \left[(x + iy)^{\bullet}\ \frac{d}{u}\right]^{\bullet}\ \frac{d}{u}$$

$$= (\ddot{x} + i\ddot{y})\ (\tfrac{d}{u})^2 - (\dot{x} + i\dot{y})\ \frac{\dot{u}}{u}\ (\tfrac{d}{u})^2 \tag{55}$$

$$\therefore\ \ddot{x} + i\ddot{y} = (\tfrac{u}{d})^2\ (x'' + iy'') + (\dot{x} + i\dot{y})\ \frac{\dot{u}}{u} \tag{56}$$

Substituting (54) and (56) in (50),

$$\frac{x'' + iy''}{d} = (J_L - i\nu J_F)\lambda + (-J_S + i\nu J_{XF})\lambda' - J_{N_\epsilon}\lambda_\epsilon\ e^{i\theta} - \frac{igd}{u^2} - \frac{a_C d}{u^2} \tag{57}$$

Integrating[1]

$$\frac{x + iy}{d} = \frac{x_o + iy_o}{d} + \frac{x_o' + iy_o'}{d}p + \int_o^p \int_o^p (J_L - i\nu J_F)\lambda\ dp\ dp + \int_o^p (-J_S + i\nu J_{XF})(\lambda - \lambda_o)dp$$

$$- J_{N_\epsilon}\lambda_\epsilon \int_o^p \int_o^p e^{i\theta}\ dp\ dp - \frac{iy_G}{d} - \frac{(x_C + iy_C)}{d} \tag{58}$$

where $y_G = gd^2 \int_o^p \int_o^p u^{-2}\ dp\ dp$

$$x_C + iy_C = \int_o^p \int_o^p a_C d^2 u^{-2}\ dp\ dp$$

If we assume that the drag coefficient is constant,

$$u^{-2} = u_o^{-2}\ e^{2J_D p} \tag{59}$$

$$\therefore\ y_G = gd^2 u_o^{-2}\left[\frac{e^{2J_D p} - 2J_D p - 1}{(2J_D)^2}\right] \doteq gd^2 u_o^{-2}\left[\frac{p^2}{2} + \frac{J_D p^3}{3} + \frac{J_D^{\,2} p^4}{6}\right] \tag{60}$$

---

1 The variation of $\nu$ in the product $\nu J_{XF}$ was neglected.

It now remains to compute the Coriolis acceleration $a_C$. It is shown in most mechanics texts that the vector $\vec{a}_C = 2\,\vec{\Omega} \times \vec{u}$ where $\vec{\Omega}$ is the angular velocity of moving axes with respect to an inertia set of axes and $\vec{u}$ is the missile's velocity with respect to the moving axes. $\vec{\Omega}$ is therefore a vector directed along the earth's axis of rotation with magnitude, $\Omega$, of $2\pi$ radians per day or $7.272 \times 10^{-5}$ rad/sec. We now introduce an auxiliary coordinate system formed by the unit vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, located at the gun and so orientated that $\vec{e}_2$ points up and $\vec{e}_3$ points north along the meridian of longitude. If $\psi$ is the angle between $\vec{e}_3$ and the ranges' z-axes for which positive orientation is defined to be from $\vec{e}_3$ to $\vec{e}_1$ and the unit vectors along the ranges' axes are denoted by $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ respectively,

$$\begin{aligned}
\vec{e}_2 &= \vec{e}_y \\
\vec{e}_3 &= \vec{e}_z \cos\psi - \vec{e}_x \sin\psi \\
\vec{e}_1 &= \vec{e}_2 \times \vec{e}_3 = \vec{e}_x \cos\psi + \vec{e}_z \sin\psi
\end{aligned} \tag{61}$$

Now $\vec{u} \doteq u\,\vec{e}_z$ (62)

and $\vec{\Omega} = \Omega\left[\vec{e}_3 \cos\theta + \vec{e}_2 \sin\theta\right]$ (63)

where $\theta$ is the latitude taken as positive for the Northern hemisphere

$$\begin{aligned}
\therefore\ \vec{a}_C &= 2\,\vec{\Omega} \times \vec{u} \\
&= 2\,\Omega\left[(\vec{e}_z \cos\psi - \vec{e}_x \sin\psi)\cos\theta + \vec{e}_y(\sin\theta)\right] \times u\vec{e}_z \\
&= 2u\,\Omega\left(\sin\psi\cos\theta\,\vec{e}_y + \sin\theta\,\vec{e}_x\right)
\end{aligned} \tag{64}$$

Using our complex notation,[1]

$$a_C = 2u\,\Omega(\sin\theta + i\cos\theta\sin\psi) \tag{65}$$

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[1] For Aberdeen Proving Ground $\theta = 39^\circ\,26'$. The ranges are so orientated that $\psi$ is 270° for the Aerodynamics Range and 168° for the Transonic Range.

$$\therefore \; x_C + iy_C = 2\,\Omega\, d^2(\sin\theta + i\cos\theta\sin\psi)\int_0^p\int_0^p u^{-1}\,dp\,dp$$

$$= \Omega\, u_o^{-1}\, d^2\,(\sin\theta + i\cos\theta\sin\psi)\left(p^2 + \frac{J_D p^3}{3} + \frac{J_D^2 p^4}{12}\right) \qquad (66)$$

Evaluation of the quartic terms shows that they may be neglected in Eqs. (60) and (66). It remains now to consider the three integrals involving aerodynamic coefficients and yawing motion. We will first consider the epicyclic yawing missile and then the more general tricyclic yawing missile.

Epicyclic Swerving Motion ($K_3 = K_{N_\epsilon}\lambda_\epsilon = 0$)

If we insert eq. (36) in eq. (58); assume $\phi_1'' = \phi_2'' = 0$, which implies $\nu' = 0$ and integrate, the result may be written as

$$\frac{x + x_C + i(y + y_C + y_G)}{d} = P + Qp + R_1K_1\, e^{(-\alpha_1 + i\phi_{10}')p} + R_2K_2\, e^{(-\alpha_2 + i\phi_{20}')p} + \frac{(x_D + iy_D)}{d} \qquad (67)$$

where P and Q are complex constants depending on initial conditions[1]

$$R_i = R_{i1} + iR_{i2}$$

$$R_{i1} = a_{i1}\, J_L + b_{i1}J_F + c_{i1}\, J_S + d_{i1}\, J_{XF}$$

$$= \frac{-(\phi_{io}'^2 - \alpha_i^2)}{(\phi_{io}'^2 + \alpha_i^2)^2}J_L + \frac{2\nu\,\alpha_i\,\phi_{io}'}{(\phi_{io}'^2 + \alpha_i^2)^2}\,J_F$$

$$+ \frac{\alpha_i}{\phi_{io}'^2 + \alpha_i^2}\,J_S + \frac{\nu\,\phi_{io}'}{\phi_{io}'^2 + \alpha_i^2}\,J_{XF}$$

$$R_{i2} = a_{i2}\, J_L + b_{i2}\, J_F + c_{i2}\, J_S + d_{i2}\, J_{XF}$$

$$= \frac{2\phi_{io}'\,\alpha_i}{(\phi_{io}'^2 + \alpha_i^2)^2}\,J_L + \frac{\nu(\phi_{io}'^2 - \alpha_i^2)}{(\phi_{io}'^2 + \alpha_i^2)^2}\,J_F$$

1 The lower limits of the integrals of Eq. (58) together with $\lambda_o$ are absorbed by P and Q.

$$+ \frac{\phi'_{io}}{\phi'^2_{io} + \alpha_i^2} J_S - \frac{\alpha_i \nu}{(\phi'^2_{io} + \alpha_i^2)} J_{XF}$$

$$\frac{x_D + iy_D}{d} = \int_0^p \int_0^p (J_L - i\nu J_F) \lambda_R \, dp \, dp + \int_0^p (-J_S + i\nu J_{XF}) \lambda_R \, dp \text{ (Drift)}$$

If we insert the approximation $-\frac{gd\bar{\nu}}{Mu^2}$ for $\lambda_R$, neglect the very small single integral and remember that for constant $\hat{D}$ and $J_D$, $\nu = \nu_o e^{\hat{D}p}$ and $u = u_o e^{-J_D p}$, the drift components may be computed to be [1]

$$\frac{x_D}{d} = -J_L \int_0^p \int_0^p \frac{gd\bar{\nu}}{Mu^2} \, dp \, dp = -J_L \frac{gd\bar{\nu}_o}{Mu_o^2} \int_0^p \int_0^p e^{(\hat{D} + 2J_D)p} \, dp \, dp \doteq -J_L \frac{gd\bar{\nu}_o}{Mu_o^2} \left(\frac{p^2}{2} + \frac{\hat{D}+2J_D}{6} p^3\right)$$

$$\frac{y_D}{d} = J_F \int_0^p \int_0^p \frac{B}{A} \frac{gd\bar{\nu}^2}{Mu^2} \, dp \, dp = -J_F \frac{gd\bar{\nu}_o^2 B}{Mu_o^2 A} \int_0^p \int_0^p e^{2(\hat{D}+J_D)p} \, dp \, dp \doteq J_F \frac{gd\bar{\nu}_o^2}{Mu_o^2} \frac{B}{A} \left(\frac{p^2}{2} + \frac{\hat{D}+J_D}{3} p^3\right) \tag{68}$$

If $J_L$ and $J_F$ are estimated, $x_D$ and $y_D$ may be computed and subtracted from x and y respectively. This then makes Eq. (67) a linear equation in eight unknowns to be fitted to the measured values of x and y at each station. This may be done by the standard least squares technique.[2] $J_L$ and $J_F$ may be computed from $R_{ij}$, inserted in the formulas for $x_D$ and $y_D$ and the process iterated. An iteration, however, is not usually necessary.

---

1 In almost all firings where drift is measurable, $\hat{D} = J_D - k_2^{-2} J_A \sim J_D$. Use of this approximation will simplify (68).

2 In this calculation the $i\phi'_{io}p$ terms which appear in the exponentials are replaced by $i(\phi'_{io}p + \phi''_i \frac{p^2}{2})$. This inconsistency in the handling of the $\phi'_i$'s was felt to yield a better approximation to the true motion. See page 635 of [1] or page 20 of [2].

For missiles possessing a fairly high rate of roll, $\phi_1'$ is usually rather large and for these missiles the size of its associated swerving motion $|R_1 K_1|$ is quite small and poorly determined. But a knowledge of $R_2$ alone puts us in the position of having to solve two equations in four unknowns! The situation is improved by the fact the $d_{21}$ and $d_{22}$ are quite small and hence $J_{XF}$ can be neglected. Since $c_{21}$ and $c_{22}$ are usually small, $J_S$ can also be neglected. In those cases where $c_{22}$ is not small enough to neglect $J_S$, an estimate of $J_S$ is inserted. The remaining set of two equations in two unknowns is then solved to yield values of $K_L$ and $K_F$.

For missiles which have been launched with no roll an important simplification is possible. For these missiles it can be seen from Eqs. (35) that $\phi_1' = -\phi_2'$ and $\alpha_1 = \alpha_2$. The additional assumption $M' = 0$ provides the result that $\phi_1'' = \phi_2'' = 0$. Insertion of these relations in the definitions of $R_{ij}$ now results in

$$R_1 = \bar{R}_2 \tag{69}$$

$$\text{or} \quad R_{11} = R_{21} = -\frac{(\phi_{10}'^2 - \alpha_1^2)}{(\phi_{10}'^2 + \alpha_1^2)^2} J_L + \frac{\alpha_1}{\phi_{10}'^2 + \alpha_1^2} J_S$$

$$R_{12} = -R_{22} = \frac{2\phi_{10}' \alpha_1}{(\phi_{10}'^2 + \alpha_1^2)^2} J_L + \frac{\phi_{10}'}{\phi_{10}'^2 + \alpha_1^2} J_S \tag{70}$$

By means of Eq. (69) the swerve curve fitting of Eq. (67) is now reduced to one involving six unknowns.[1] Both the six and eight unknowns swerve reductions have been coded for the Bell Relay Computer although only the eight unknowns swerve reduction has been coded for the Edvac.

Although the eight unknowns swerve reduction just described has been employed successfully for some years, it possesses three undesirable features. They are:

1. The reduction is so formulated that the contribution of the higher frequency mode of the yawing motion to the swerving motion has to be neglected.

2. In order to correct for drift it is necessary to guess a $K_L$ and a $K_F$. It would be desirable to use drift in the evaluation of $K_L$ and $K_F$.

3. The effect of $\phi_1'' \neq 0$ is handled in an inconsistent manner.

1 Note also that for no spin $x_D = y_D = 0$.

It was decided to eliminate features (1) and (2) by placing equation (67) in a form in which the aerodynamic coefficients themselves and not the intermediate $R_i$'s are the unknowns. In doing this it was further decided to neglect the effect of $K_{XF}$. Finally the inconsistent treatment of non-zero $\phi_i''$'s was corrected by means of an asymptotic series expansion of the swerve integrals.

In order to obtain this expansion we will consider the integral $\int e^{\beta} dp$ where $\beta''$ is constant. If this integral is integrated by parts,

$$\int e^{\beta} dp = \int \frac{e^{\beta}\beta'}{\beta'} dp$$

$$= \frac{e^{\beta}}{\beta'} + \beta'' \int \frac{e^{\beta}\beta'}{\beta'^3} dp$$

$$= \frac{e^{\beta}}{\beta'} + \frac{\beta'' e^{\beta}}{\beta'^3} + 3\beta''^2 \int \frac{e^{\beta}}{\beta'^4} dp \doteq e^{\beta}(\beta'^{-1} + \beta'' \beta'^{-3}) \qquad (71)$$

The neglection of the remainder term is valid for small $\frac{\beta''}{\beta'^2}$ since its magnitude is approximately $3 \left| \frac{\beta''}{\beta'^2} \right|^2 \left| \frac{e^{\beta}}{\beta'} \right|$. An examination of the process of generation of this series shows that the magnitude of the remainder term at the nth step is approximately $1 \cdot 3 \cdot 5 \cdots\cdots (2n-1) \left| \frac{\beta''}{\beta'^2} \right|^n \left| \frac{e^{\beta}}{\beta'} \right|$. From this we see that for $\left| \frac{\beta''}{\beta'^2} \right| < 1$ the remainder will first decrease and then increase without bound. This property is a characteristic of an asymptotic series.

For the small $\frac{\beta''}{\beta'^2}$ usually encountered, however, Eq. (71) is a good approximation to the integral. If we integrate (71) again, neglecting squares of $\frac{\beta''}{\beta'^2}$,

$$\iint e^{\beta} dp dp = \frac{e^{\beta}}{\beta'^2} + \frac{\beta'' e^{\beta}}{\beta'^4} + 2\beta'' \int \frac{e^{\beta}}{\beta'^3} dp$$

$$\doteq e^{\beta} \left(\beta'^{-2} + 3\beta'' \beta'^{-4}\right) \qquad (72)$$

By means of (71) and (72), (58) can be written as:

$$\frac{x + x_C + i(y + y_G + y_C)}{d} = P + Qp + J_L S_1 + \nu_o J_F S_2 + J_S S_3 \tag{73}$$

where[1]

$$S_1 = K_1 e^{\beta_1}\left[\beta_1'^{-2} + 3\beta''_1 \beta_1'^{-4}\right] + K_2 e^{\beta_2}\left[\beta_2'^{-2} + 3\beta''_2 \beta_2'^{-4}\right] + \frac{x_D}{J_L d}$$

$$S_2 = K_1 e^{\hat\beta_1}\left[\hat\beta_1'^{-2} + 3\beta''_1 \hat\beta_1'^{-4}\right] + K_2 e^{\hat\beta_2}\left[\hat\beta_2'^{-2} + 3\beta''_2 \hat\beta_2'^{-4}\right] + \frac{y_D}{\nu_o J_F d}$$

$$S_3 = K_1 e^{\beta_1}\left[\beta_1'^{-1} + \beta''_1 \beta_1'^{-3}\right] + K_2 e^{\beta_2}\left[\beta_2'^{-1} + \beta''_2 \beta_2'^{-3}\right]$$

$$\beta_i = -\alpha_i p + i(\phi'_{io} p + \phi''_i \frac{p^2}{2})$$

$$\hat\beta_i = (\hat D - \alpha_i)p + i(\phi'_{io} p + \phi''_i \frac{p^2}{2})$$

The definitions of the $S_i$'s together with equation (68) show that they are known functions of p. Eq. (73) is then a linear equation in seven unknowns and can be fitted by the usual least squares method. This seven unknowns swerve reduction will be referred to as the LFS swerve reduction. For certain rounds either $J_F$ or $J_S$ can be omitted thereby providing us with LS and LF swerve reductions respectively. The LF reduction has been particularly successful and is the standard epicyclic swerve reduction coded for the Ordvac. The LS swerve which for zero spin is equivalent to the six unknowns swerve has been coded for the Ordvac also. Both the LFS and LF swerve reduction have been programmed for Bell Relay Computer.

Tricyclic Swerving Motion

If the complete Eqs. (36) and (58) are combined, the following expanded form of Eq. (67) for non-zero constant spin may be obtained.

$$\frac{x + x_C + x_D + i(y + y_G + y_C + y_D)}{d} = P + Qp + R_1K_1 e^{(-\alpha_1 + i\phi'_{10})p} + R_2K_2 e^{(-\alpha_2 + i\phi'_{20})p} + R_3K_3 \frac{e^{i\theta}}{\nu^2} \tag{74}$$

---

1 From Eq. (68) we see that $\frac{x_D}{J_L d}$ is $-\frac{gd\bar\nu_o}{Mu_o^2}\left(\frac{p^2}{2} + \frac{\hat D + 2J_D}{6}p^3\right)$ and, hence, is independent of $J_L$. This term comes from $\iint \lambda_R \, dp \, dp$. A similar observation applies to $\frac{y_D}{\nu_o J_F d}$.

where $R_3 = -(J_L - \frac{J_{N_\epsilon}\lambda_\epsilon}{K_3}) + i\nu\ (J_S + J_F) + \nu^2\ J_{XF}$

Eq. (74) presents a routine ten unknowns curve fitting problem for least squares. This has been coded for the Bell Relay Computer.[1]

For zero spin Eq. (74) is invalid. A second reduction (L-L) has been coded for the Bell Relay Computer for use on tricyclic rounds with zero or near zero spin, ($|\nu| < 10^{-4}$). This reduction was derived on the additional simplifying assumption that all the aerodynamic coefficients other than $J_L$ may be neglected. The near zero constant spin assumption allows the replacement of $e^{i\theta}$ in Eq. (36) by

$$e^{i\theta_o}\ (e^{i\nu p}) = e^{i\theta_o}\left[1 - \frac{\nu^2\ p^2}{2!} + \frac{\nu^4\ p^4}{4!} + i(\nu p - \frac{(\nu p)^3}{3!} + \frac{(\nu p)^5}{5!})\right] .$$

For the integration, only the first term of Eq. (72) is used and the drift terms are omitted.

$$\therefore\ \frac{x + x_G + i(y + y_G + y_C)}{d} = P + Qp + J_{L_1}\ S_1^* + J_{L_2}\ S_4 \qquad (75)$$

where

$$S_1^* = \frac{K_1\ e^{\beta_1}}{\beta_1'^{\,2}} + \frac{K_2\ e^{\beta_2}}{\beta_2'^{\,2}} \doteq S_1$$

$$S_4 = \frac{K_3\ e^{i\theta_o}}{-\nu^2}\left[e^{i\nu p} - (1 + i\nu p)\right]$$

$$\doteq -K_3\ e^{i\theta_o}\left[-\frac{p^2}{2!} + \frac{\nu^2 p^4}{4!} - \frac{\nu^4 p^6}{6!} + i(\frac{-\nu p^3}{3!} + \frac{\nu^3 p^5}{5!} - \frac{\nu^5 p^7}{7!})\right];\ |\nu| < 10^{-3}$$

Since it was felt that $J_{L_2} K_3$ is well determined by the swerving motion, both $J_L$'s are solved for in the hope that their ratio would provide a correction to $K_3$.

---

1 Here again $i\phi'_o p$ terms appearing in the exponentials are replaced by $i(\phi'_{io}p + \phi''_i \frac{p^2}{2})$ .

Finally we will describe the tricyclic swerve reduction which is being coded for the Ordvac. This reduction will be based on the ideas which were developed in the various swerve reductions already discussed. It has been decided to reduce the four degrees of freedom expressed by $R_1$ and $R_2$ of Eq. (74) to two, namely $J_L$ and $J_S$, revise the tricyclic term so that zero spin is no longer a special case and incorporate the full corrections of non-zero $\phi_i''$'s. With these aims in mind the following equation may be derived.

$$\frac{x + x_C + x_D + i(y + y_G + y_C)}{d} = P + Qp + J_L S_1 + J_S S_3 + R_3 S_4 \qquad (76)$$

where the symbols are already defined for Eqs. (73) - (75).

It can be shown that the major component of the imaginary part of $R_3$ arises from $J_{N_\epsilon}$. Both this coefficient and $J_S$ were omitted from the L-L swerve. The reduction will probably be expanded to handle slowly varying spin by the insertion of $\nu_o p + \frac{\nu_o'}{2} p^2$ for $\nu$ in $S_4$. This will be useful for finned missiles over the portion of trajectory for which $\nu$ is within 5% of steady state.[1] This now completes our discussion of the individual reduction processes.

It is interesting to consider the relative time requirements of the computing machines for the various problems. This comparison is provided by Table I. Although the Bell Computer is rather slow, it has the important feature of being practically error-free.

---

1 The advantages of numerical integration for the calculation of $S_4$ are being considered. This is feasible when the problem is done on the high speed Ordvac.

TABLE I

| A Bell Computer | | Time in Minutes* |
|---|---|---|
| 1. Drag Reduction | 4 unknowns | 15-25 |
| 2. Yaw Drag Reduction | 5 unknowns | 90-150 |
| 3. Roll Differential Corr. | 4 unknowns | 60-90 |
| 4. Epicyclic Yaw Diff. Corr. | 6 unknowns | 7n + 40 |
| 5. Epicyclic Yaw Diff. Corr. | 8 unknowns | 9n + 50 |
| 6. Epicyclic Yaw Diff. Corr. | 10 unknowns | |
| a. First iteration | | 12n + 65 |
| b. Later iteration | | 10n + 65 |
| 7. Tricyclic Yaw Diff. Corr. | 8 unknowns | 14n + 55 |
| 8. Tricyclic Yaw Diff. Corr. | 10 unknowns | |
| a. First iteration | | 16n + 50 |
| b. Later iterations | | 14n + 50 |
| 9. Epicyclic Swerve | 6 unknowns | 6n + 30 |
| 10. Epicyclic Swerve | 8 unknowns | |
| a. Short** | | 7n + 70 |
| b. Long | | 12n + 70 |
| 11. LF Swerve | 6 unknowns | |
| a. Short | | 20n + 30 |
| b. Long | | 25n + 30 |
| 12. LFS Swerve | 7 unknowns | |
| a. Short | | 23n + 40 |
| b. Long | | 28n + 40 |
| 13. Tricyclic Swerve | 10 unknowns | 6n + 40 |
| 14. Tricyclic Swerve (L-L) | 6 unknowns | 10n + 30 |

* All times are for one run only. n is the number of stations in the round.

** Short swerve reductions can be performed only when the yaw reduction has been done on the Bell Computer.

TABLE I (Con't)

B Edvac

| | | | Minutes |
|---|---|---|---|
| 1. | Epicyclic Yaw | 10 unknowns | 2.5 |
| 2. | Epicyclic Swerve | 8 unknowns | 6.6 |

Due to delays of tape handling the usual time for two iterations of yaw plus swerve is about 15 min.

C Ordvac***

| | | | |
|---|---|---|---|
| 1. | Center of mass reduction | | $(4n + 30)$ sec. |
| 2. | Drag Reduction | 4 unknowns | 2 min. (max.) |
| 3. | Roll Diff. Corr. | 4 unknowns | 3 min. |
| 4. | Epicyclic Yaw Corr. | 10 unknowns | 5 min. |
| 5. | Epicyclic Swerve (L-F) | 6 unknowns | 4 min. |

The usual time for two iterations of yaw plus swerve is about twelve minutes.

*** All reductions on the Ordvac employ the binary point routines described in [12] .

## CRITERIA FOR QUALITY OF RESULTS

In conclusion we will consider the various ways in which the quality of the results of the reductions described above may be estimated. There are essentially three ways by means of which this can be done.

First the sizes of the residuals of each fitted curve may be compared with the accuracy of the corresponding instrumentation. This comparison will reveal computational errors and identify rounds for which the particular theory is inappropriate but will tell very little about the accuracy of determination for individual coefficients.

The second method describes the internal consistency of the measurements and involves the computation of the statistically defined standard errors[1] of each aerodynamic coefficient. Briefly, if $X_i$ is a parameter of one of the least squares fits described above, it can be shown that $\epsilon_{X_i}$, the standard error in $X_i$, may be computed by the relation:

$$\epsilon_{X_i} = \sqrt{A_{ii}}\ \epsilon \tag{77}$$

where $\epsilon = \sqrt{\dfrac{\sum R^2}{n-r}}$ (standard error of fit)

R residuals of fit

n number of measurements

r number of unknowns

$A_{ij}$, elements of the inverse of the matrix formed by the coefficients of the least squares normal equations.

If an aerodynamic coefficient is a function $f(X_1, X_2)$ of two of these parameters the following relation is used to calculate the coefficient's standard error:

$$\epsilon^2_{f(X_1,\,X_2)} = \left(\frac{\partial f}{\partial X_1}\epsilon_{X_1}\right)^2 + \left(\frac{\partial f}{\partial X_2}\epsilon_{X_2}\right)^2 + 2\left(\frac{\partial f}{\partial X_1}\right)\left(\frac{\partial f}{\partial X_2}\right)\epsilon_{X_1 X_2} \tag{78}$$

where $\epsilon_{X_1 X_2} = A_{12}\epsilon^2 = A_{21}\epsilon^2$

The generalizations of Eq. (78) are clear.

---

1 The commonly used probable errors may be calculated by multiplication of the standard errors by .6745.

Experience has shown that the relative magnitudes of the errors for the rounds of a program are quite useful. For example the usual size of errors in $K_M$ for bodies of revolution is about .03. Rounds which have errors in $K_M$ which are double this are inspected with some care. The absolute magnitude of statistical errors is usually much smaller than the round to round scatter and so is given little weight.

The third and most effective approach examines the effect which the aerodynamic coefficient in question has on the motion and determines criteria for the motion which must be satisfied in order that this effect be measurable. For example, in order to determinate the drag coefficient the observed deceleration of the missile must be measurable in terms of timing accuracies of 5/8 μ sec., and distance accuracy of either .015" for the aerodynamics range or .01' for the transonic range. This distinction is quite important. If an extremely heavy missile were fired with no yawing motion and a drag reduction performed, the residuals of fit would quite probably be very good. The standard error for $K_D$ could be quite reasonable and yet the $K_D$ would be meaningless due to the fact that the actual deceleration would be smaller than could be measured accurately. A second example could be constructed from the roll reduction of a missile whose rolling velocity is within 5% of steady state throughout its observed flight. The fit could be excellent and the calculated value of $C_l$ completely worthless!

Since the analyses of the drag and roll reductions have already been made in [4] and [10] respectively, we will only discuss criteria for the yaw and swerve reductions. Experience with the graphical portion of the yaw reductions shows us the $K_{10}$ and $K_{20}$ must be of reasonable size. Since $K_M$ depends primarily on $\phi_1'$ and $\phi_2'$ and these are quite well determined for $K_{io}$'s over five times the experimental error of .001 rad, the criterion for $K_M$ is that $K_{10} \geqslant .005$ and $K_{20} \geqslant .005$. If a spin reduction has been performed, only one of the $K_{io}$'s need be larger than .005 while $\phi_i'$ for the other is computed from Eq. (36). Since H and T depend on the shrinking or growing of the two arms, the requirement on the amplitudes of the two modes must be strengthened. Experience has determined that the lower bound of .007 for the amplitudes of both modes is sufficient for reasonable determination of H and T.

The criteria for swerve reductions requires a little more algebraic work. First we must define measures of the effect of $K_L$, $K_F$ and $K_S$ on the swerving motion. In Eq. (67) only $R_2$ need be considered for spinning bodies of revolution. For this case the lift swerve $(R_2)_L$ and Magnus swerve $(R_2)_F$ are defined by the relations:

$$(R_2)_L = J_L K_{20} \sqrt{a_{21}^2 + a_{22}^2} \tag{79}$$

$$(R_2)_F = J_F K_{20} \sqrt{b_{21}^2 + b_{22}^2} \tag{80}$$

We then require that $(R_2)_L$ or $(R_2)_F$ be five times the experimental accuracy in calibers in order that the corresponding coefficient be determined. (This transverse distance accuracy for the Transonic Range is .01 ft. while for the Aerodynamics Range it is .015 in.) For the six unknowns swerve described by Eqs. (69) and (70), the following definitions are natural:

$$(R)_L = J_L \sqrt{K_{10}^2 + K_{20}^2} \cdot \sqrt{a_{21}^2 + a_{22}^2} \qquad (81)$$

$$(R)_S = J_S \sqrt{K_{10}^2 + K_{20}^2} \cdot \sqrt{c_{21}^2 + c_{22}^2} \qquad (82)$$

Even simpler definitions are possible for Eq. (73)

$$S_L = J_L \sqrt{\left(|S_1|^2\right)_{\text{average}}} = J_L \sqrt{\frac{\Sigma |S_1|^2}{n}} \qquad (83)$$

$$S_F = \nu_o J_F \sqrt{\left(|S_2|^2\right)_{\text{average}}} = \nu_o J_F \sqrt{\frac{\Sigma |S_2|^2}{n}} \qquad (84)$$

$$S_S = J_S \sqrt{\left(|S_3|^2\right)_{\text{average}}} = J_S \sqrt{\frac{\Sigma |S_3|^2}{n}} \qquad (85)$$

where summations are over all stations and n is the number of stations[1]. The criteria for $(R)_L$, $(R)_S$, $S_L$, $S_F$, $S_S$ is the same as that for $(R_2)_L$ and $(R_2)_F$. The tricyclic swerve may be treated in the same manner.

[1] Since $\sum |S_i|^2$ appear as a coefficient in the normal equations, definitions (83) - (85) are especially convenient.

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## TABLE OF SYMBOLS[1]

| | |
|---|---|
| $A$ | Axial moment of inertia; constant in Roll Eq. (31) |
| $A_{ij}$ | Element of inverse of normal matrix for least squares [Eq. (77)] |
| $B$ | Transverse moment of inertia; constant in Roll [Eq. (31)] |
| $C$ | Constant in modified roll equation [Eq. (31')] |
| $D = J_D - \frac{md^2}{A} J_A$ | |
| $D_o = D$ at $M = M_o$ | |
| $\hat{D} = D + \frac{D_o}{\nu}$ | |
| $E$ | Constant in modified roll equation [Eq. (31')] |
| $(F_1, F_2, F_3)$ | Aerodynamic Force vector in missile coordinate system |
| $(F_X, F_Y, F_Z)$ | Aerodynamic Force vector in range coordinate system |
| $G = \frac{igd}{u^2}\left[J_D + k_2^{-2} J_H - i\bar{\nu}\right]$ | |
| $H = J_L - J_D + \frac{md^2}{B} J_H$ | |
| $I_1(p)$ | Integral defined for Eq. (26) |
| $I_2(p)$ | Integral defined for Eq. (26) |
| $J_i = \frac{\rho d^3}{m} K_i$ | |
| $J_\epsilon = \left[i\nu(1-\frac{A}{B}) J_{N_\epsilon} - k_2^{-2} J_{M_\epsilon}\right]\lambda_\epsilon$ | |
| $K_A$ | Spin deceleration moment coefficient |
| $K_{A_o}$ | Spin producing moment coefficient |
| $K_D$ | Drag force coefficient (trajectory drag) |
| $K_{DA}$ | Axial drag force coefficient |
| $K_{D_{\delta^2}} = \frac{\partial K_D}{\partial \delta^2}$ | Yaw drag coefficient |
| $K_{D_M} = \frac{\partial K_D}{\partial M}$ | Slope of Mach number drag curve |
| $K_F$ | Magnus force coefficient |

1 A few symbols unfortunately have two meanings. These are separated by semi-colons. The definition which applies may easily be determined by the context.

TABLE OF SYMBOLS (Con't)

| | |
|---|---|
| $K_H$ | Damping moment coefficient |
| $K_L$ | Lift force coefficient |
| $K_M$ | Overturning (righting) moment coefficient |
| $K_{M_\epsilon}$ | Asymmetrical moment coefficient |
| $K_N$ | Normal force coefficient |
| $K_{N_\epsilon}$ | Asymmetrical normal force coefficient |
| $K_S$ | Damping force coefficient |
| $K_T$ | Magnus moment coefficient |
| $K_{XF}$ | Magnus force coefficient due to cross spin |
| $K_{XT}$ | Magnus moment coefficient due to cross spin |
| $K_{io}$ | Amplitude of the ith mode of yawing motion |
| $K_i = K_{io}e^{i\phi_{io}}$ | |
| L | Distance from missile's reference point to center of mass |
| $M = k_2^{-2}J_M$; or Mach number | |
| $(M_1, M_2, M_3)$ | Aerodynamic moment vector |
| P | Constant in Eq. (58); Pressure |
| $P_W$ | Vapor pressure of water |
| $P_d$ | Pressure of dry air |
| Q | Constant in Eq. (58) |
| R | Residuals |
| $R_{ij}$ | Expression defined for Eq. (67) |
| $R_i = R_{i1} + iR_{i2}$ | |
| $(R_2)_L, (R)_L$ | Radii of lift swerve motion for Eqs. (67) and (69) respectively |
| $(R_2)_F$ | Radius of Magnus swerving motion for Eq. (67) |

TABLE OF SYMBOLS (Con't)

| | |
|---|---|
| $(R)_S$ | Radius of damping force swerving motion for Eq. (69) |
| $R_X$, $R_Y$ | Constants in Eqs. (6) and (7) |
| $S_i$ | Quantities appearing in Eq. (73) |
| $S_F$ | Radius of Magnus swerving motion [Eq. (84)] |
| $S_L$ | Radius of lift swerving motion [Eq. (83)] |
| $S_F$ | Radius of damping force swerving motion [Eq. (85)] |
| $T = J_L - k_1^{-2} J_T$; Absolute temperature | |
| $V_S$ | Velocity of sound |
| $V_{SD}$ | Velocity of sound in dry air |
| a,b | Constants defined by $D=(a + bM)^{-1}$ |
| $a_C$ | Coriolis acceleration |
| $a_i$ | Coefficient of drag equation [Eq. (21)] |
| $a_{ij}$ | Quantities defined for Eq. (67) |
| $b_{ij}$ | Quantities defined for Eq. (67) |
| $(c_1, c_2)$ | x and y coordinates of horizontal spark |
| $(c_1', c_2')$ | x and y coordinates of vertical spark |
| $c_{ij}$ | Quantities defined for Eq. (67) |
| d | Diameter |
| $d_{ij}$ | Quantities defined for Eq. (67) |
| $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ | Unit vectors along 1,2,3 axis on earth's surface |
| $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ | Unit vectors along x,y,z axis located on the range |
| g | Acceleration due to gravity |
| $k_1 = \sqrt{\frac{A}{md^2}}$ | Axial radius of gyration in calibers |

| | |
|---|---|
| $k_2 = \sqrt{\frac{B}{md^2}}$ | Transverse radius of gyration in calibers |
| $m$ | Mass |
| $\bar{m} = \bar{\nu}^2 - 4M - H^2$ | |
| $(m,n,p)$ | Direction cosines of missile's axis |
| $n$ | Number of observations |
| $p = \frac{z}{d}$ | |
| $r$ | Number of unknowns; quantity defined in Eq. (29) |
| $s$ | Steady state spin |
| $s_o$, $s'$ | Constants defined by $k_1^{-2} J_{A_o} = -D(s_o + s'p)$ |
| $u$ | Missile's velocity |
| $u_1$ | Axial component of missile's velocity |
| $x,y,z$ | Positional coordinates in range system |
| $(x_H,0,z_H)$, $(0,y_V,z_V)$ | Coordinates of plate measurements |
| $(x_R,y_R,z_R)$ | Space coordinate of reference point on missile |
| $(x_{cm},y_{cm},z_{cm})$ | Space coordinate of center of mass of missile |
| $x_D,y_D$ | x and y components of drift |
| $x_C,y_C$ | x and y components of Coriolis deflection |
| $y_G$ | Gravity drop |
| $\alpha_i$ | Exponential damping coefficient of ith mode |
| $\beta$ | Exponent defined for Eq. (71) |
| $\beta_i = -\alpha_i p + i(\phi'_{io} p + \phi''_i \frac{p^2}{2})$ | |
| $\hat{\beta}_i = \beta_i + \hat{D}p$ | |
| $\gamma = \frac{-J\eta}{1+a\eta bM_o}$ | |
| $\gamma = \phi_{20} - \phi_{10} - 2\eta$ [Eq. (29)] | |

TABLE OF SYMBOLS (Con't)

| | |
|---|---|
| $\delta = \lvert\lambda\rvert$ | magnitude of yaw |
| $\overline{\delta^2}$ | Mean squared magnitude of yaw |
| $\epsilon = \sqrt{\frac{\Sigma R^2}{n - r}}$ | Standard error of fit |
| $\epsilon_i$ | Standard error in i |
| $\epsilon_I$ | Perturbation term arising from slowly varying spin [Eq. (35)] |
| $\eta = \arccos. - \frac{(\alpha_1 + \alpha_2)}{r}$ | [Eq. (29)] |
| $\theta$ | Roll angle; latitude |
| $\lambda = \lambda_H + i\,\lambda_V = \lambda_2 + i\,\lambda_3$ | |
| $\lambda_H, \lambda_V, \lambda_2, \lambda_3$ | Components of yaw in radians along horizontal, vertical 1 and 2 axis respectively |
| $\lambda_\epsilon$ | Magnitude of asymmetry angle |
| $\lambda_R$ | Yaw of repose |
| $\mu = \frac{\omega_2 d + i\,\omega_3 d}{u_1}$ | nondimensional cross angular velocity |
| $\nu = \frac{\omega_1 d}{u_1}$ | nondimensional spin |
| $\bar{\nu} = \frac{A}{B}\nu$ | |
| $\rho = \rho_a + \rho_w$ | air density |
| $\rho_a, \rho_w$ | "partial densities" defined after Eq. (2) |
| $\phi_i = \phi_{io} + \phi'_{io}p + \phi''_i \frac{p^2}{2}$ | orientation of i, the mode, in complex yaw plane |
| $\psi$ | Azimuth |
| $\Omega$ | Magnitude of earth's angular velocity |
| $(\omega_1, \omega_2, \omega_3)$ | Missile's angular velocity |
| $(\ )^{\cdot}$ | Derivative with respect to time |
| $(\ )'$ | Derivative with respect to p |
| $\overline{(\ )}$ | Average or conjugate of complex quantity |

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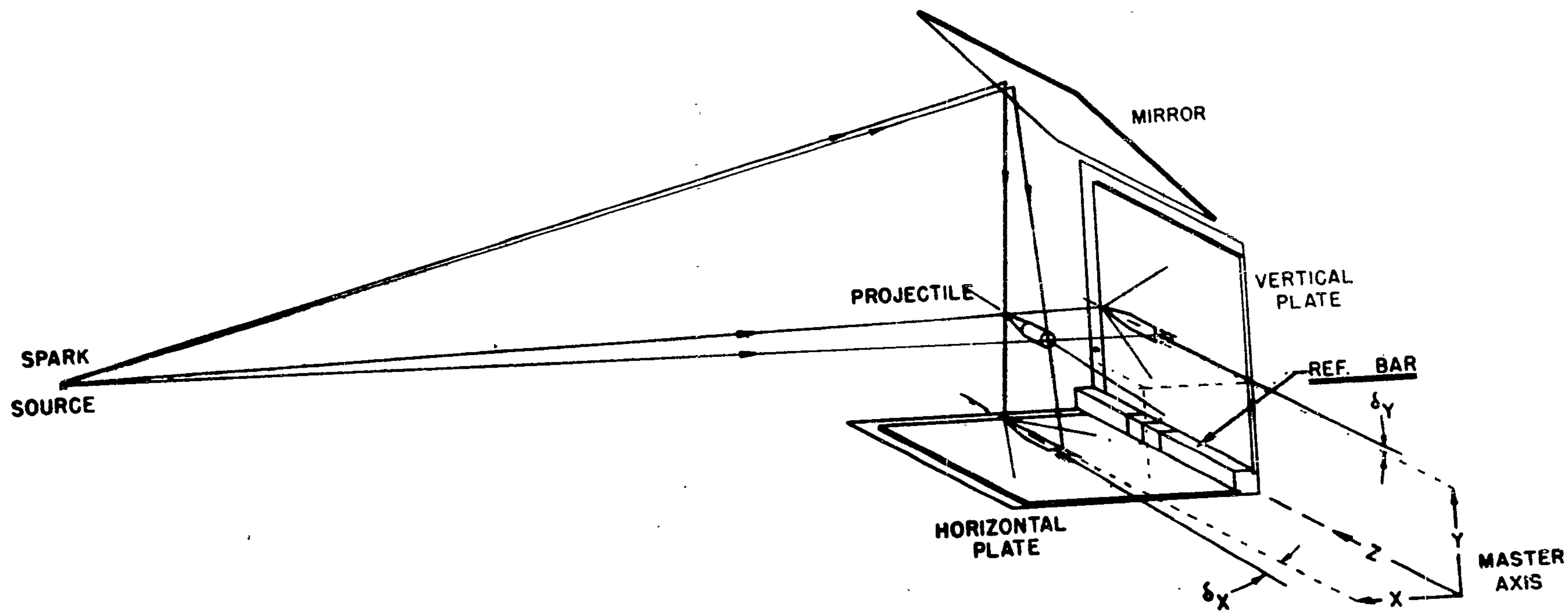
FIG. 1

AERODYNAMICS (SMALL) SPARK RANGE LAYOUT



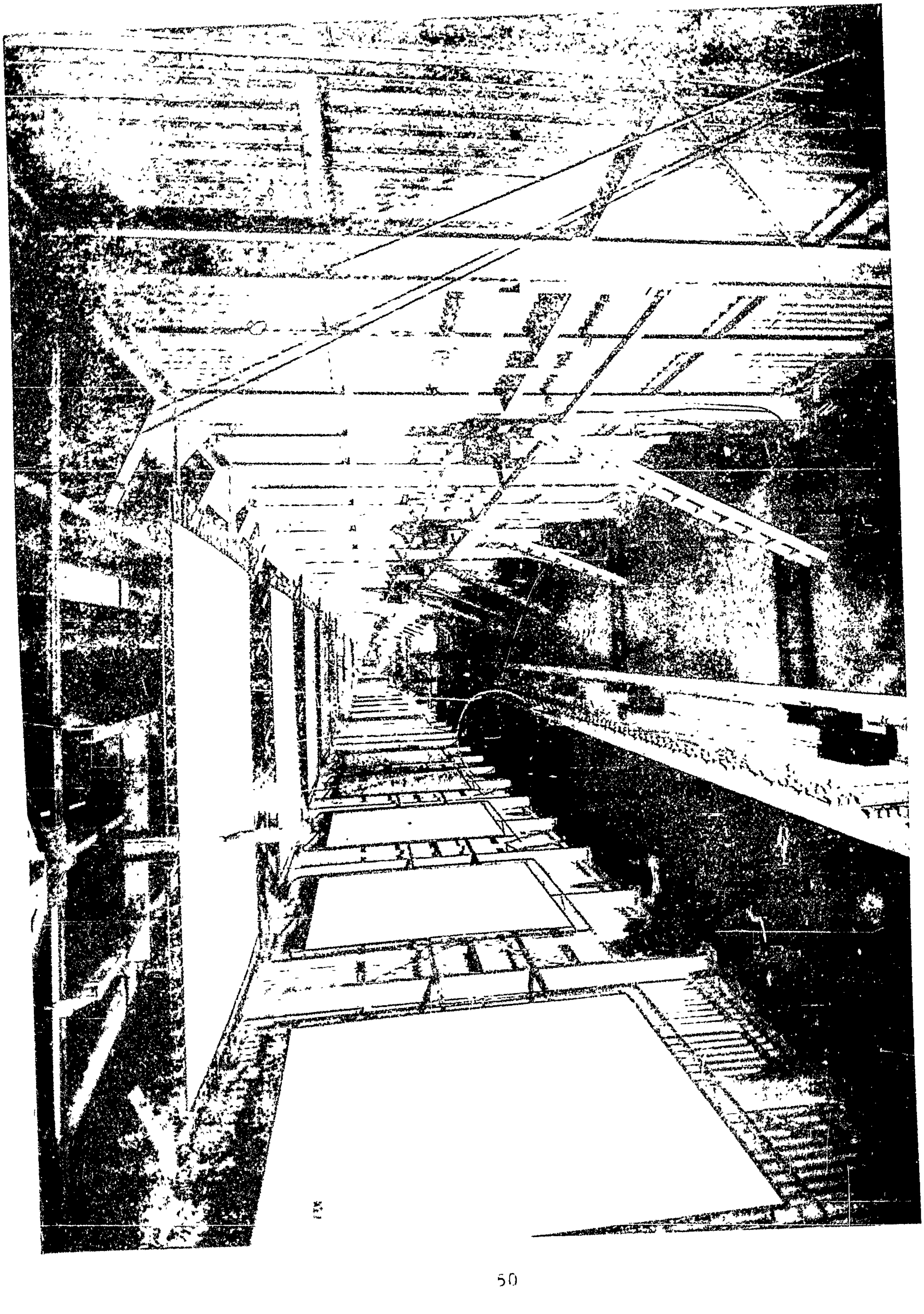


FIG. 3

TRANSONIC RANGE SPARK STATION LAYOUT

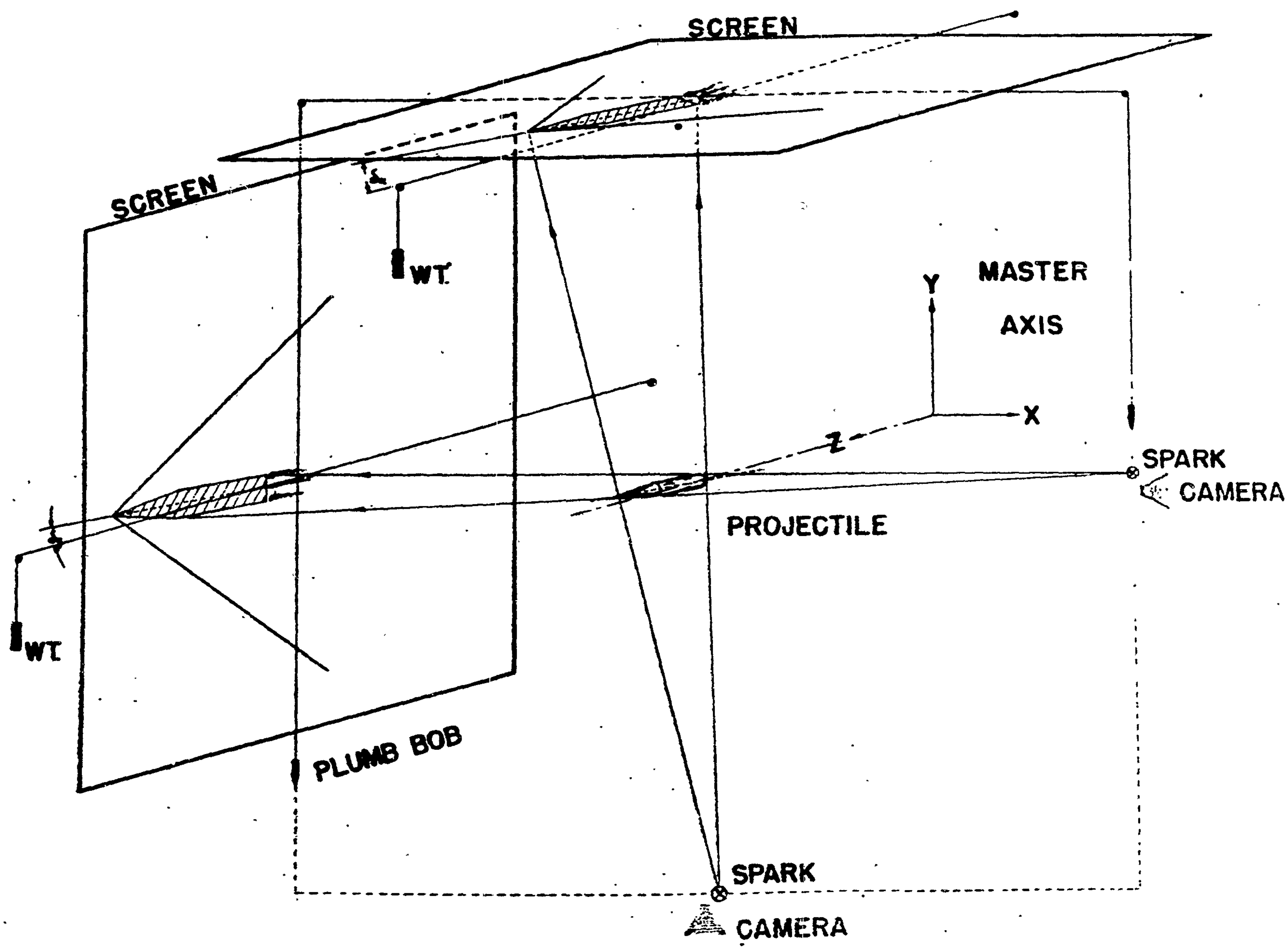



Fig. 4

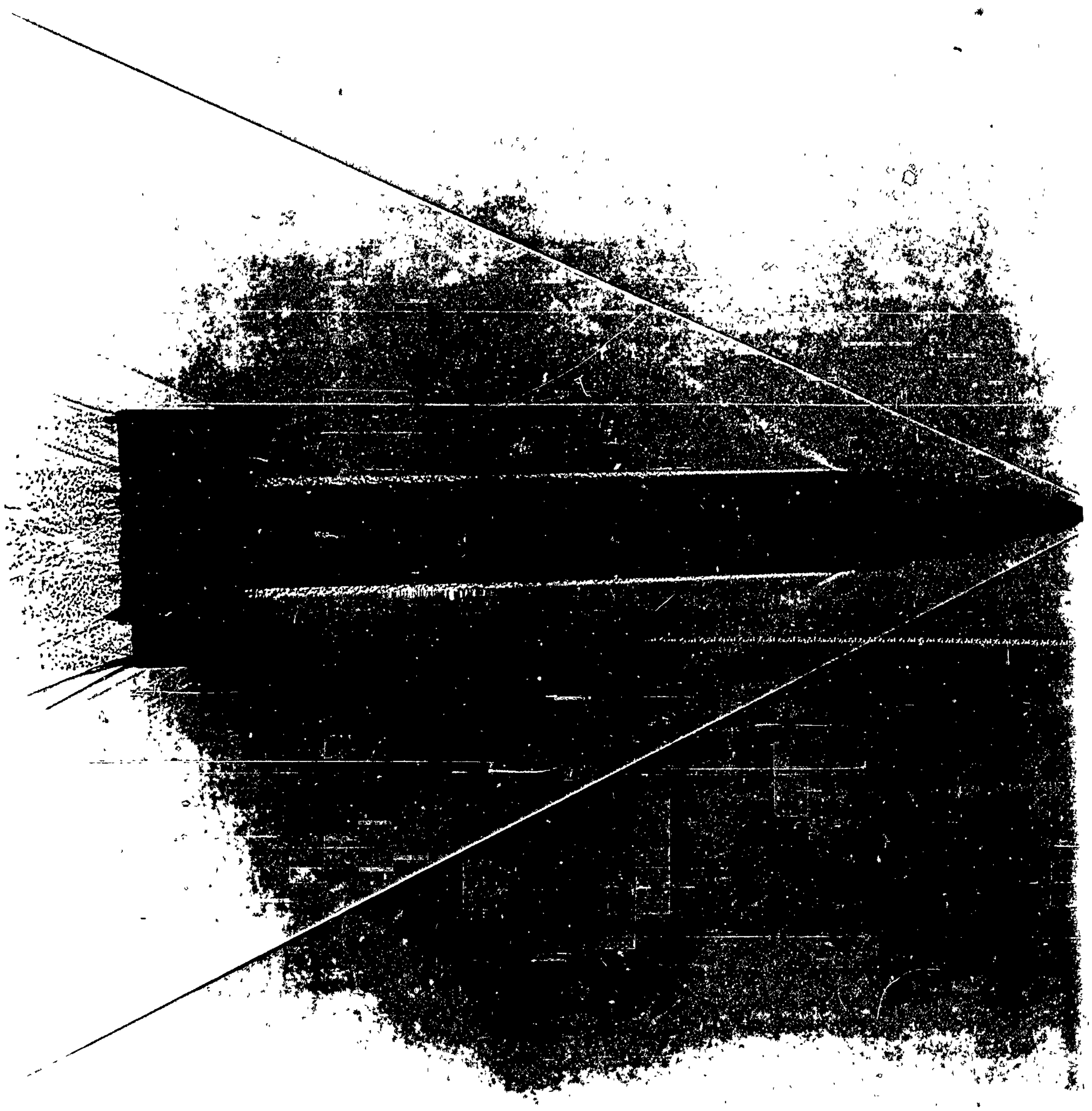


FIG. 5

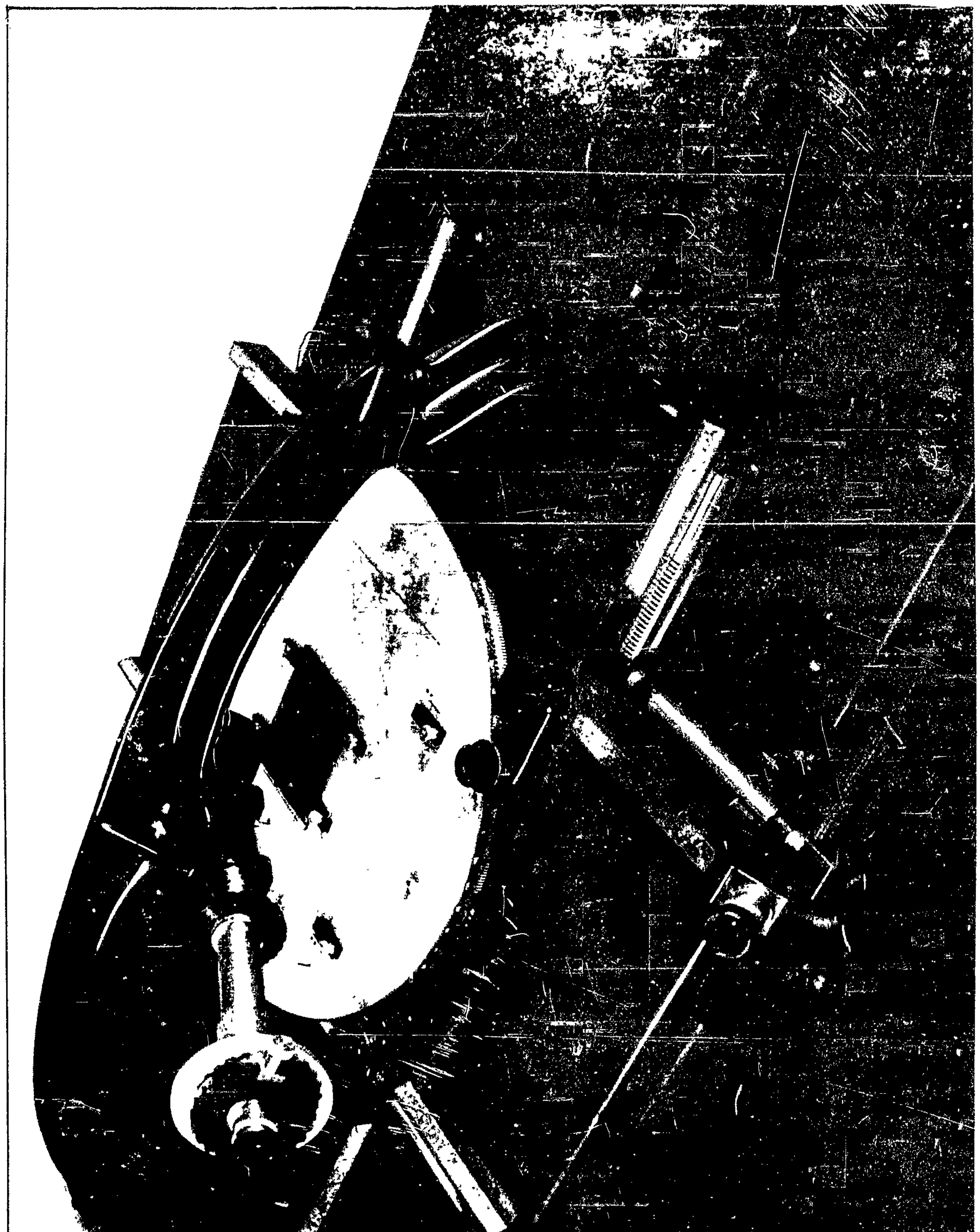


FIG. 6

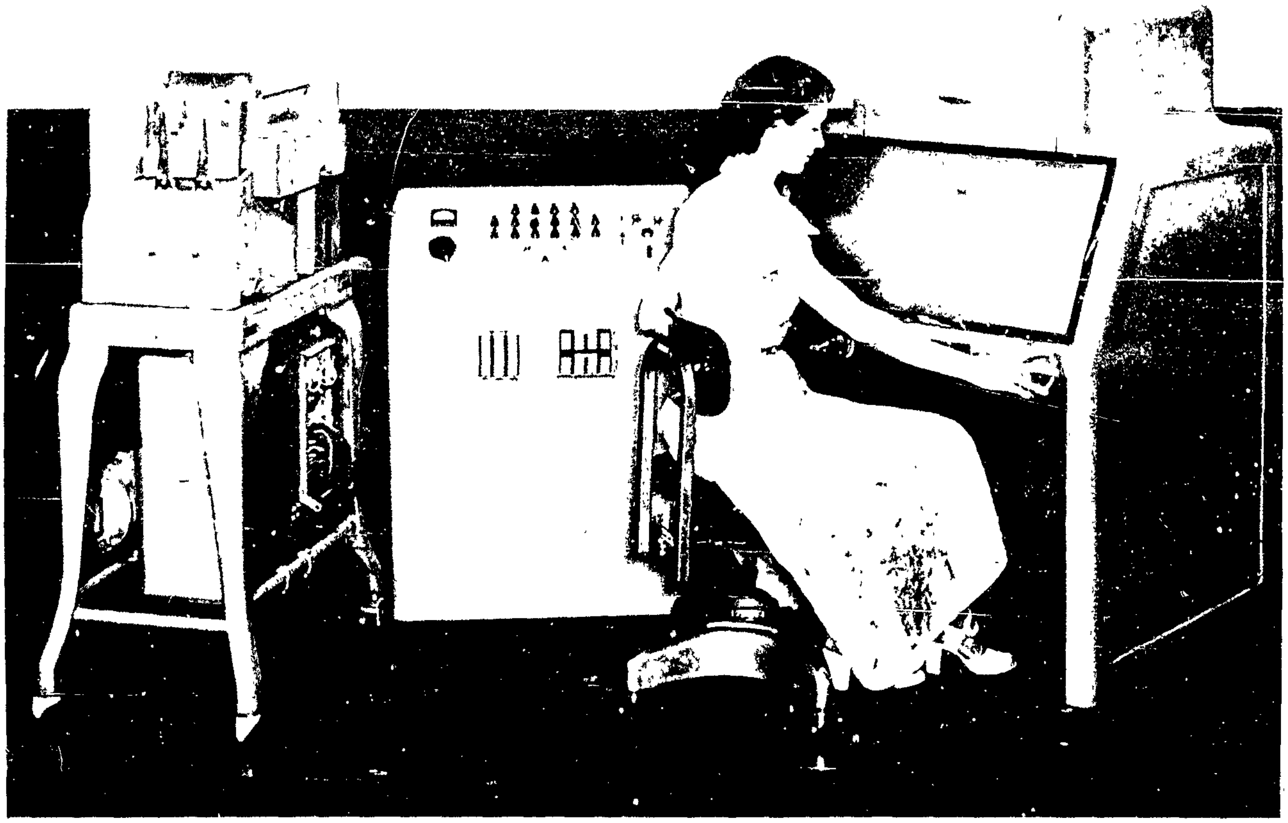


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FIG. 7

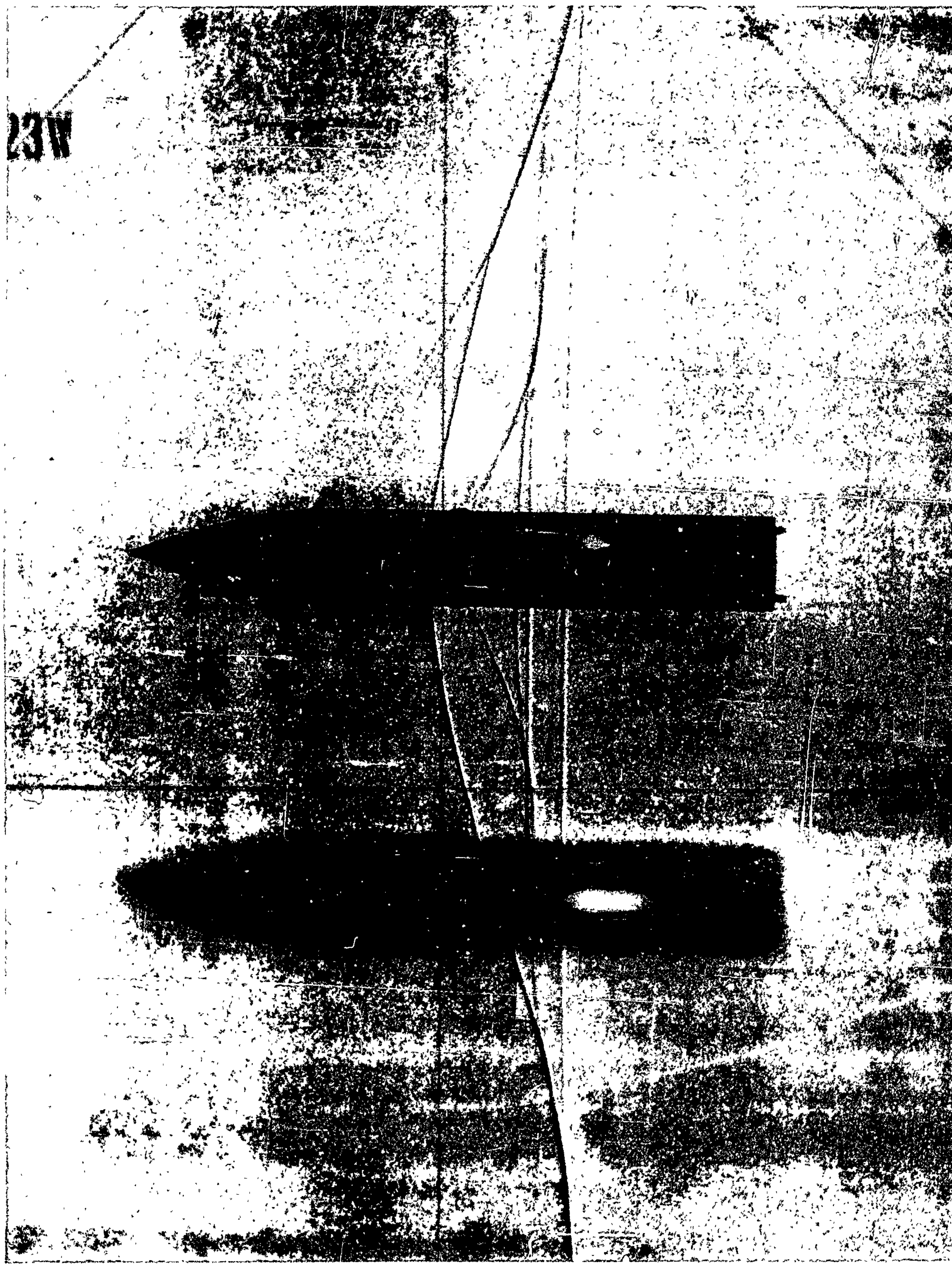
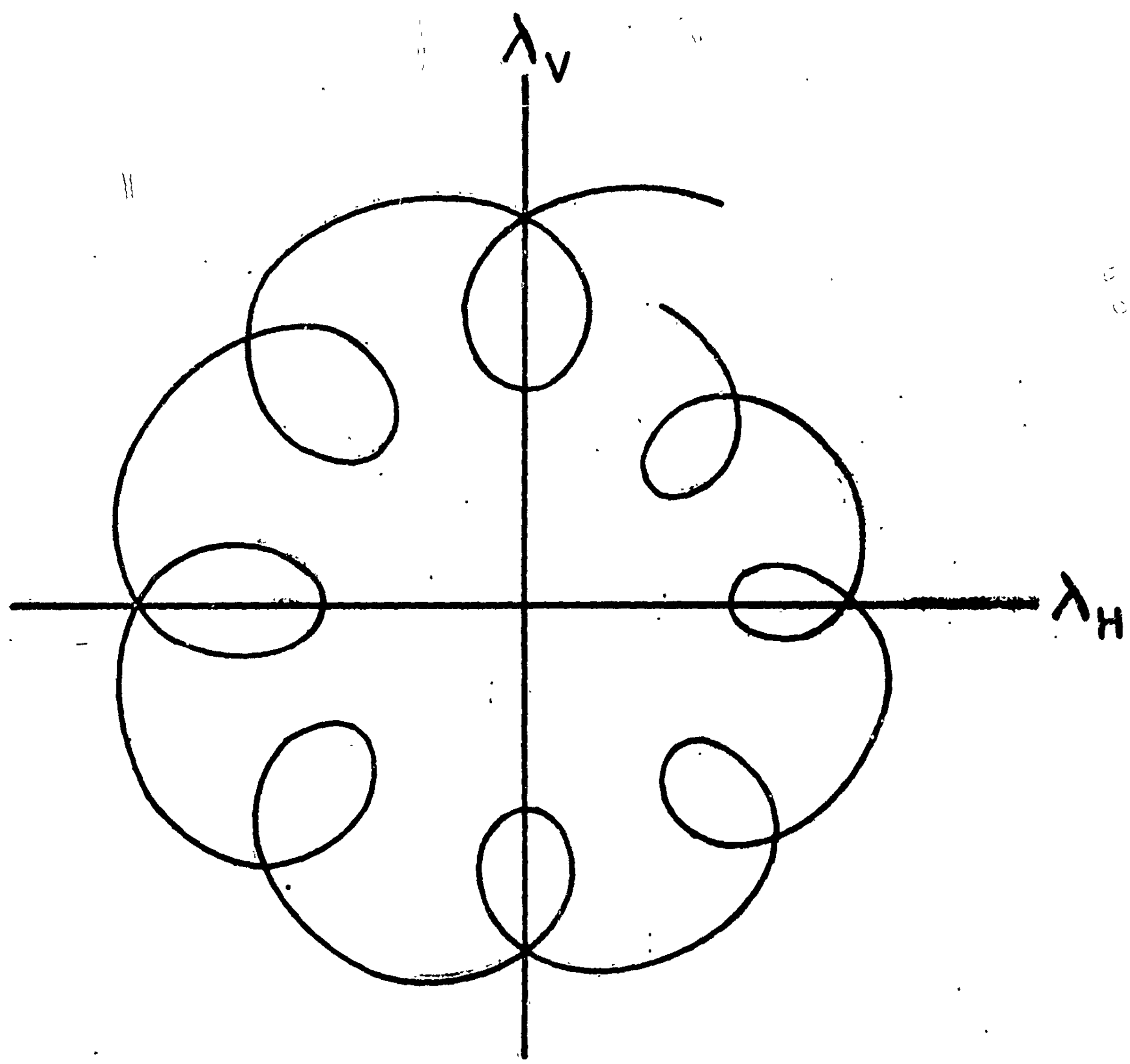
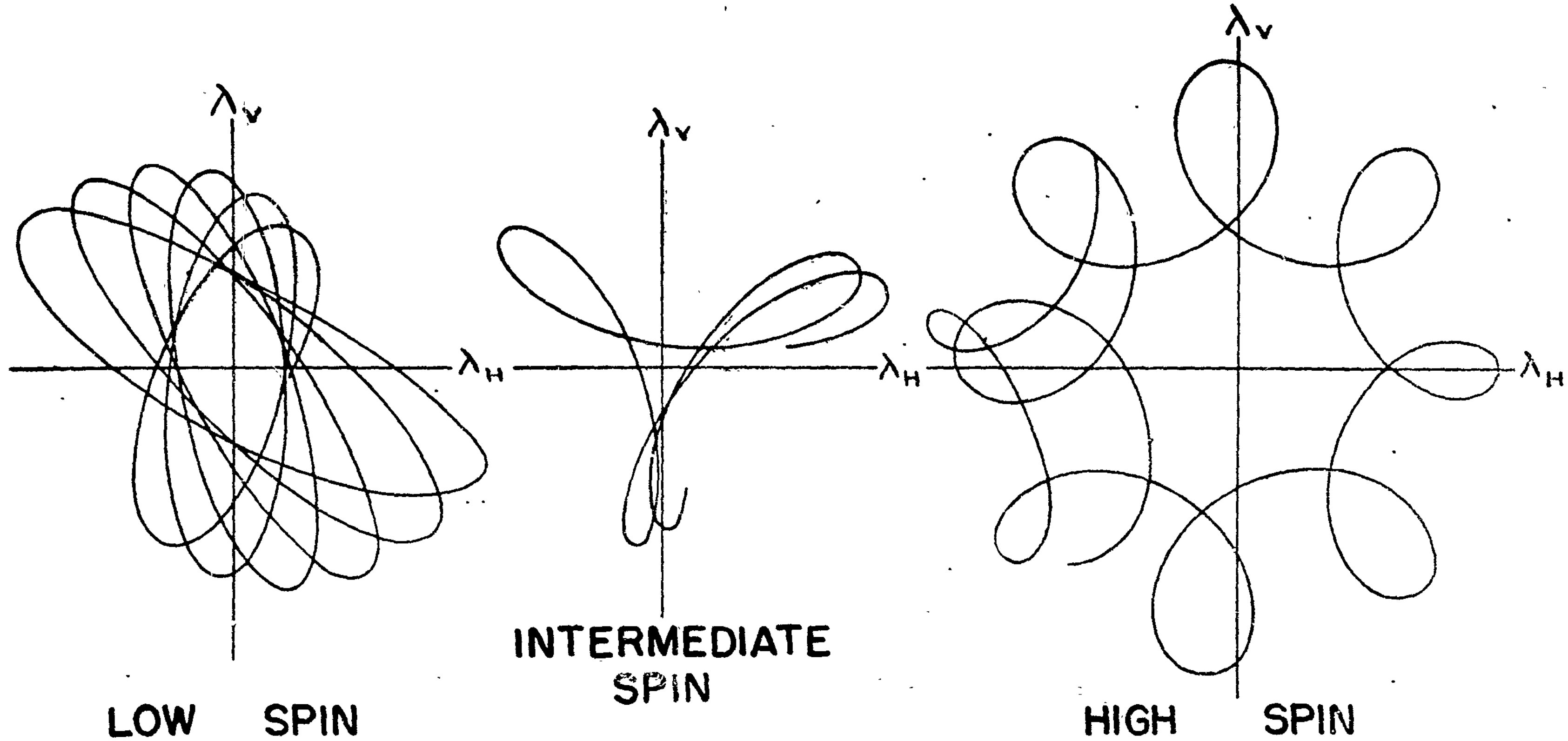


FIG. 8




TYPICAL FREE FLIGHT YAWING MOTION
OF A RAPIDLY SPINNING BODY
OF REVOLUTION

FIG. 9




TYPICAL FREE FLIGHT YAWING MOTION OF SPINNING FINNED MISSILES

FIG. 10

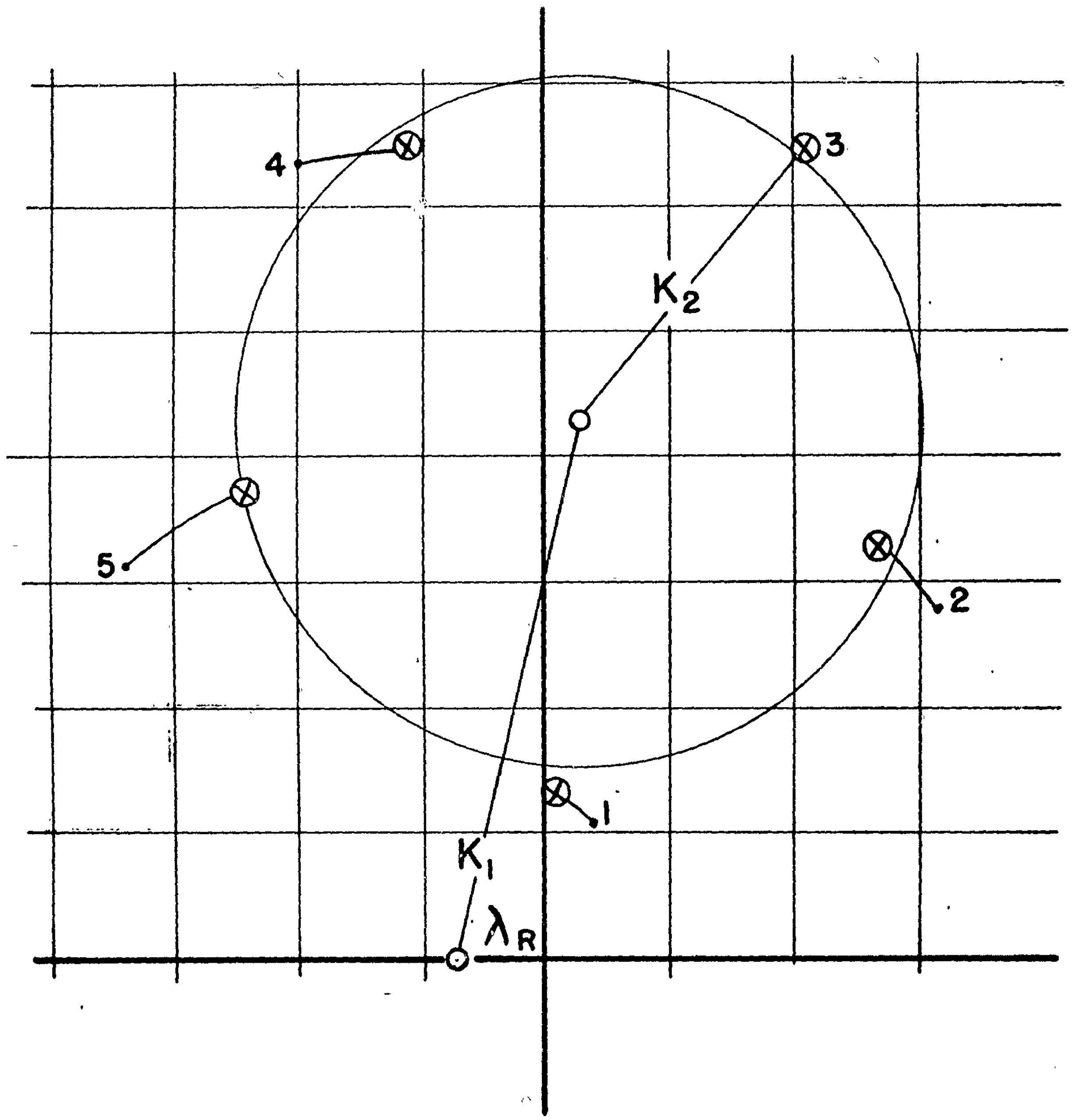
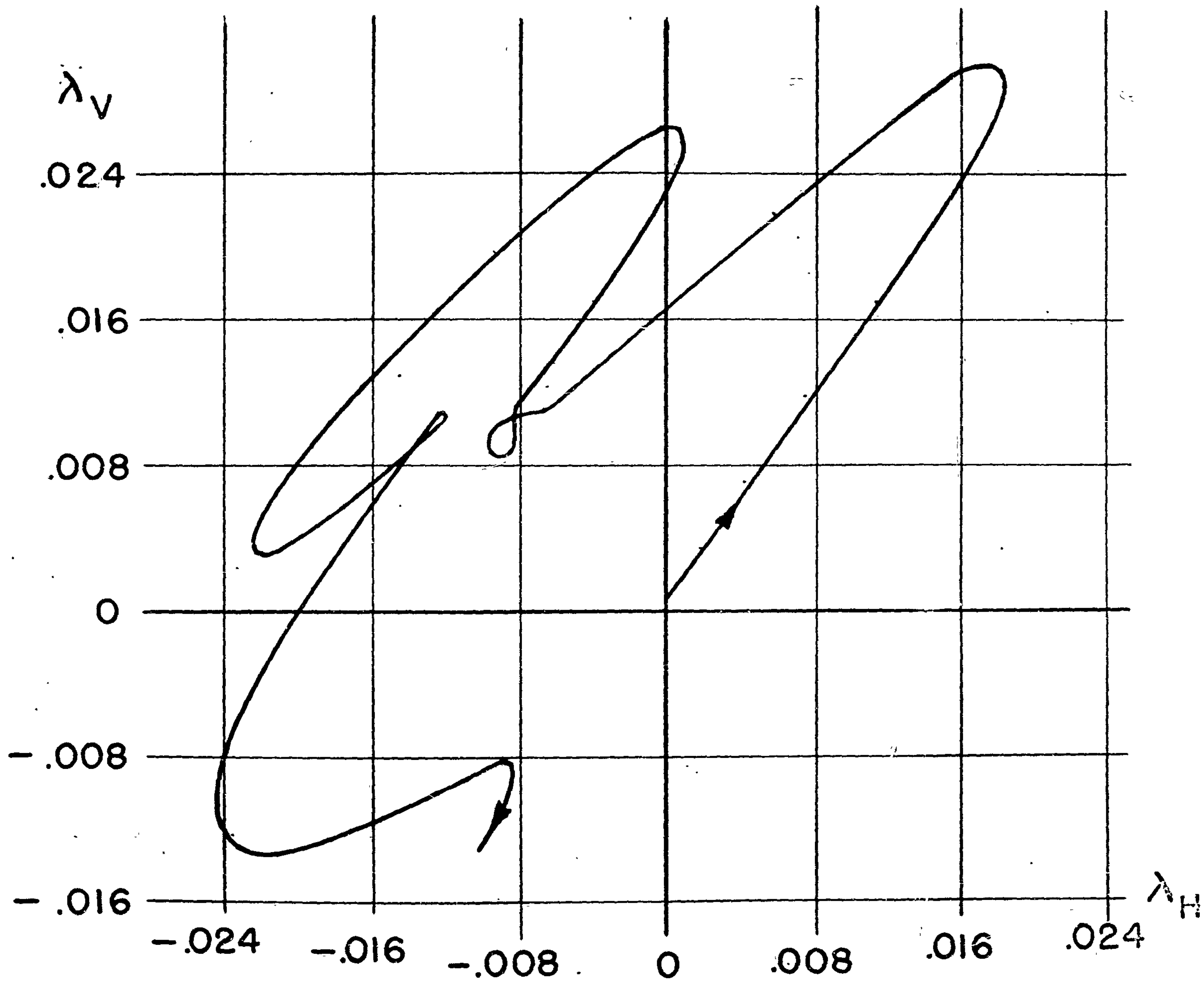
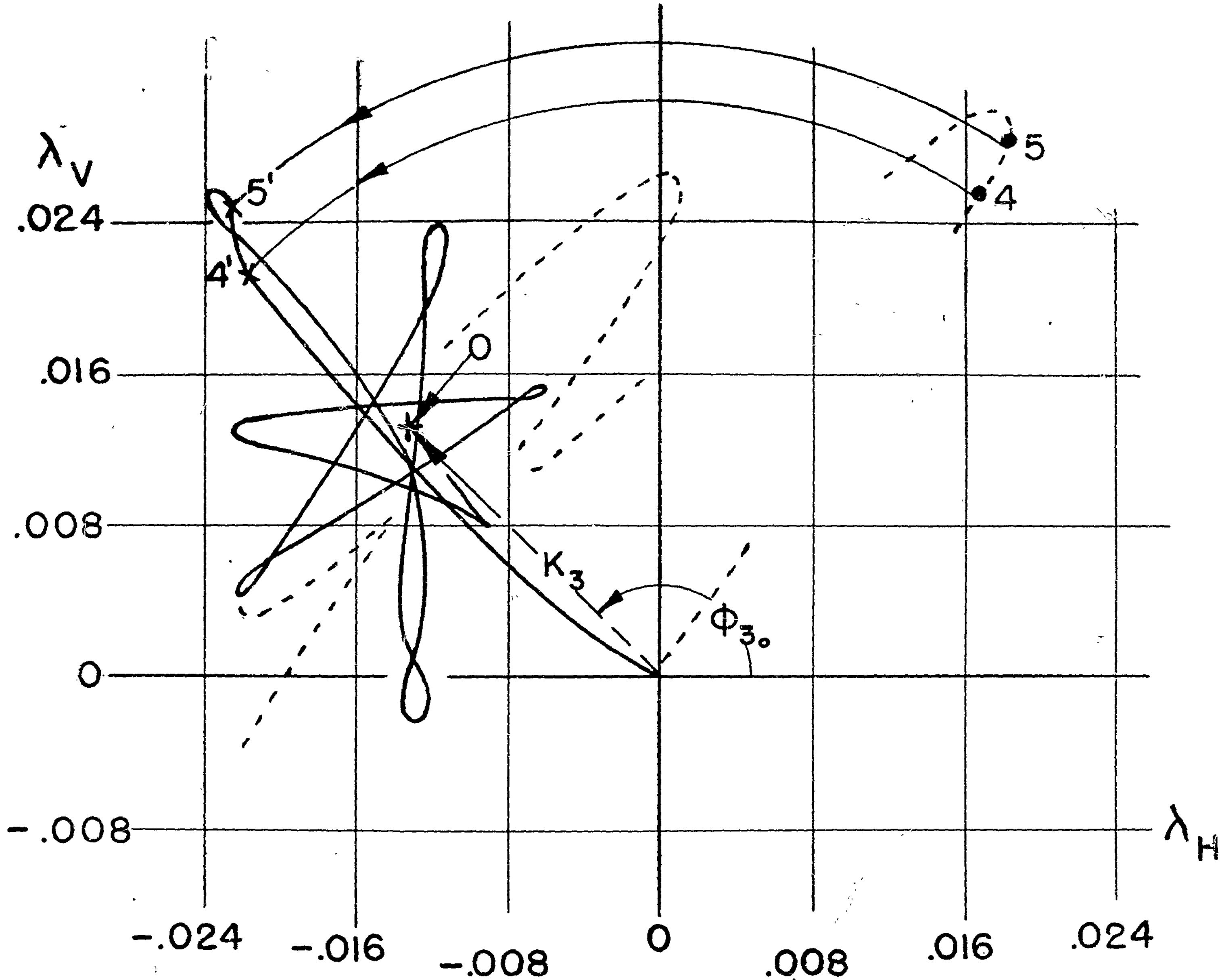



Fig. 11



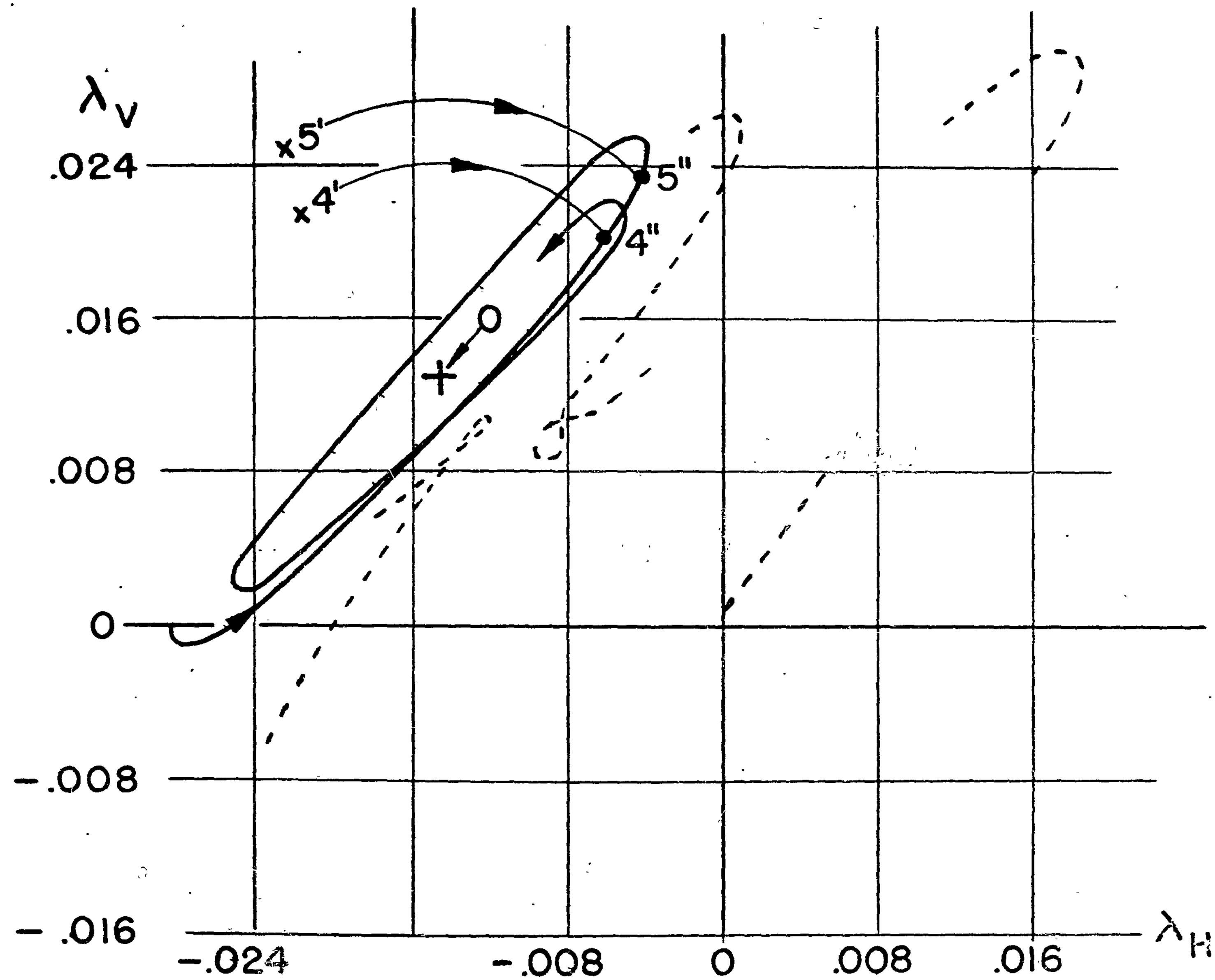

Original Asymmetrical Motion

Fig. 12




Points Rotated Through Roll Angle

Fig. 13




Points Rotated Back With Respect To New Origin

Fig. 14

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